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RADIATIVE TRANSFER - AN OUTLINE OF PROF. SUBRAMANYAN CHANDRASEKHAR's CONTRIBUTIONS [1]

S. V. G. MENON

CONTACT INFORMATION:

e-mail: menonsvg@yahoo.co.in

e-mail: menon.svg98@gmail.com

mobile: +91 8879394488

mobile:+91 8547992824

[1]Lecture given in Indian Physics Association's colloquium, organized to felicitate Prof. Subramanyan Chandrasekhar.

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Born	October 19, 1910 Lahore, British India
Died	August 21, 1995 (aged 84) Chicago, Illinois, United States
Residence	United States
Citizenship	India (1910–1953) United States (1953–1995)
Fields	Astrophysics
Institutions	University of Chicago University of Cambridge
Alma mater	Presidency College, Madras Trinity College, Cambridge
Doctoral advisor	R.H. Fowler, Arthur Stanley Eddington
Doctoral students	Donald Edward Osterbrock,Roland Winston, F. Paul Esposi-to,Jeremiah P. Ostriker
Known for	Chandrasekhar limit
Notable awards	**Nobel Prize in Physics (1983)** Copley Medal (1984) National Medal of Science (1966) Padma Vibhushan (1968)

Radiative Transfer -

An outline of Prof. Subramanyan Chandrasekhar's contributions.

S. V. G. Menon

Introduction:

Radiative transfer theory deals with the study of transfer of radiant energy through a medium that absorbs, scatters or emits radiation. Radiation can be of electromagnetic origin – light, x-rays – or of nuclear type like, neutrons, γ-rays or α-particles.

Prof. Subramanyan Chandrasekhar made very significant and outstanding contributions to this field - which, perhaps, is one of the most important branches of theoretical Astrophysics today. He published a number of original papers, most of them in the Astrophysical Journal, during a decade starting from 1942. Most of these were really path breaking in that the whole subject was being evolved and developed with these papers. Thus he was instrumental to show examples, where, neither the author nor the journal, can anticipate the impact of a scientific work.

Prof. Chandrasekhar also wrote the first definitive monograph on the subject in 1950: ***Radiative Transfer*** [1]. Almost all his original papers are discussed in great detail in this book, and it would be futile even to attempt to describe them here.

However, a few remarks in the preface of the book seem important: "In this book I have attempted to present the subject of radiative transfer in plane parallel atmospheres as a branch of mathematical physics with its own characteristic methods and techniques. On the physical side the novelty of the methods used consists of employment of certain general principles of invariance which on the mathematical side leads to the systematic use of non-linear integral equations and the development of the theory of a special class of such equations. On these accounts the subject would seem to have an interest which is beyond that of the specialist alone: at any rate, that has been my justification for writing the book."

It is also very interesting to note his remarks[2] about the field of radiative transfer: " My research on radiative transfer gave me the most satisfaction. I worked on it for five years, and the subject, I felt, developed on its own initiative and momentum. Problems arose one by one, each more complex than the previous one, and they were solved. The whole subject attained an elegance and a beauty which I do not find to the same degree in any of my other work. And when I finally wrote the book *Radiative Transfer*, I left the area entirely. Although I could think of several problems, I did not want to spoil the coherence and beauty of the subject [*by further additions*]. Furthermore, as the subject had developed, I also had developed. It gave me for the first time a degree of self-assurance and confidence in my scientific work because here was a situation where I was not looking for problems. The subject, not easy by any standards, seemed to evolve on its own."

The subject of radiative transfer originated way back in the 18th century when physicists were concerned with the visibility of the atmospheres. It was at this time Lord Rayleigh formulated his famous scattering law of light by a particle with a specified dielectric constant. However, its proper formulation as a classical macroscopic theory was given by Arthur Schuster in a paper titled "Radiation through a foggy atmosphere" in 1905. Radiative transfer theory is based on concepts of radiation intensity, energy density, degree of polarization, etc. Interaction of radiation with matter is described on a phenomenological level in terms of scattering, absorption and emission properties of the medium. For the case of light, where frequency is about 6×10^{11} per second, intensity of radiation propagating in a certain direction can be expressed as a 'time average' of the Poynting vector of electromagnetic theory. Just like the radiation intensity, the Poynting vector also must vary slowly over a spatial region of detector resolution to use this concept. Phenomena such as interference and diffraction of light are not included in the 'classical' domain of radiative transfer theory. However, their effects can be incorporated in material properties.

When properties of the medium – which includes absorption, scattering and emission characteristics – and certain conditions regarding the incidence of radiation at the boundaries of the medium are specified, the theory determines the radiation distribution throughout the medium. Problems in atmospheric and astrophysical areas related to transfer of light provided the necessary impetus for the development of the theory. In addition to playing a great part in the physical phenomena occurring in astrophysical bodies, radiative transfer governs the light in the outer atmosphere, and hence the character of their spectra. Further, today it has a lot of relevance due to its applications in fission and fusion reactor designs.

Equations of Radiative Transfer

One of the simplest cases of radiative transfer equations is that for a plane parallel medium that reads as

$$\mu \frac{\partial}{\partial x} I_\nu(x,\mu) + K_\nu \rho\, I_\nu(x,\mu) = J_\nu = K_\nu \rho \frac{1}{2} \int_{-1}^{1} p(\mu_0)\, I_\nu(x,\mu')\, d\mu' \quad (1)$$

[2] " CHANDRA ", A Biography of S. Chandrasekhar, by K. C. Wali, The University of Chicago Press (1991), page 190.

where I_ν is the radiation intensity – of frequency ν – varying only along the x-axis, μ is the cosine of the angle between the direction of motion and the x-axis, K_ν is the mass absorption coefficient and ρ is the density of the medium. The function $p(\mu_0)$ (μ_0 is the cosine of the scattering angle), which is called the scattering law of the medium, specifies the angular distribution of the scattered radiation. While the terms on the left denote, respectively, the rates of radiation flow and its removal from the direction of motion in a stationary situation, J_ν gives the rate at which radiation is scattered into the direction of motion. It is assumed that the medium does not absorb radiation, its scattering is elastic and there are no sources inside. These generalizations can be incorporated very easily. As an example, this equation can describe the steady transfer of radiation incident on some boundary. A medium that steadily transfers the radiation incident on it is said to be in radiative equilibrium. Equations exactly of this type occur in the description of neutron and γ-ray transport in reactor and shield media [2]. Thus the relevance of radiative transfer theory to the nuclear industry is obvious.

An alternate situation is of a medium, like the stellar atmosphere, which is in local thermodynamic equilibrium. Then, on assuming that a temperature T can be specified at every point in the medium, the term J_ν can be expressed as $J_\nu = K_\nu \rho B_\nu$, where B_ν is the well known Planck function:

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Here, h and k denote, respectively, the Planck and Boltzmann constants, and c is the velocity of light. The total radiation intensity $\phi(x)$ – which can be obtained by integrating $I_\nu(x,\mu)$ over μ and ν– can then be related to the local temperature using Stefan's law: $\phi = \sigma T^4/\pi$, where σ is Stefan's constant, thereby yielding an equation for temperature. The mass absorption coefficient, K_ν, strongly depends on temperature, and hence the problem highly non-linear. Solutions to problems of this nature yield the steady temperature distribution in the stellar atmosphere, its brightness, etc.

In certain problems, as occurring in laser induced fusion concepts, the deposition of energetic radiation, which can also be energetic electrons or ions, induces local heating and hydrodynamic motion in the medium. The 'pdV' work done on the medium, due to the induced hydrodynamics, can further heat up the medium with subsequent production of thermal radiation. The transfer of the radiation so produced is of utmost importance in the design of fusion reactor systems. For example, the thermal energy and radiant energy become comparable at about 100 eV in a dilute gas with particle number density ~ 10^{19} per cm^3. Even at lower temperatures, the radiant flux can become comparable to material energy flux as the velocity of light is much larger than fluid velocity. These problems are intrinsically time dependent, and further, the equations of radiative transfer get coupled with equations of hydrodynamics. Thus the 'energy equation' of invicid hydrodynamics becomes [3]

$$\frac{\partial}{\partial t}\rho(\varepsilon+u^2/2)=-\frac{\partial}{\partial x}[\rho u(\varepsilon+u^2/2)+up+S]$$

where ε is the specific internal energy, u is fluid speed and p is fluid pressure. The radiant heat flux $S=c\,\phi$ (last term on the right) is to be obtained from the solution of the time dependent radiative transfer equation. Radiant energy and work done by radiation pressure are neglected in this equation. Having made these general remarks about the applications of radiation transfer theory, let us go back to the main theme of the lecture.

Wick-Chandrasekhar Method :

Prof. Chandrasekhar developed several new mathematical techniques for solving the radiation transport equations. With a view to have a method of sufficient generality, Prof. Chandrasekhar – independent of G. C. Wick – pioneered the use of an approximation scheme to solve radiation transfer problems. In this method the integral in Eq.(1) is approximated using Gauss's quadrature formula :

$$\int_{-1}^{1} I_v(x,\mu)\,d\mu = \sum_{j=1}^{m} a_j\, I_v(x,\mu_j)$$

where a_j's are the weights and μ_j's are the ordinates of the m^{th} order formula. Use of this formula was a generalization of an earlier work of A. Schuster, who had employed the specific case of 2^{nd} order Gauss's quadrature. With this approximation, Eq.(1) reduces to a system of differential equations for $I_v(x, \mu_j)$ for $1 \leq j \leq m$., thus bringing it to the 'standard form' for further analysis. Thus Prof. Chandrasekhar solved a variety of problems in radiation transfer theory. He writes that the aim was to have "a general method capable of disclosing unsuspected relationships between solutions to different problems". Most interestingly, the method led to the discovery of a new set of mathematical functions – the ***H-functions*** –in radiation transfer theory.

Before discussing the H-functions, let it be noted that a modified form of the Wick-Chandrasekhar method – called the discrete ordinates method – is used routinely today in almost all numerical algorithms dealing with transfer of radiation. This generalized method was developed at the Los Alamos Laboratory, by Carson and Lathrop, for solving sufficiently general problems, including different geometries, in radiation transport theory.

Chandrasekhar's H-functions :

There are two important problems in radiation transfer theory of planar media:

i) Determination of the angular distribution of emergent radiation $I_v(0, \mu)$ from a semi-infinite medium, say, extending over the positive half plane, with a constant flux of radiation flowing through it.

ii) Determination of the angular distribution $I_v(0, \mu, \mu_i)$ of reflected radiation from a semi-infinite medium when radiation is incident along a particular direction, say, at an angle specified by μ_i.

The relevance of these problems to astrophysics is, perhaps, obvious: While the first models the radiation emerging out of the outer atmosphere of star (Fig.1 a), the second mimics the reflection of incident radiation on a planetary object (Fig.1 b). Thus, the solutions to these are, respectively, known as the 'law of darkening' and the 'law of diffuse reflection'.

Chandrasekhar's method, mentioned above, showed a remarkable fact that solutions to both problems can be expressed in terms of a single function which he called the *H-function*. For the case of an isotropic scattering medium, a representation of the H-function turns out to be:

$$H(\mu) = \frac{1}{\mu_1\,\mu_2 \ldots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n-1} (1 + k_\alpha \mu)} \; ; \qquad \sum_{j=1}^{n} \frac{a_j}{(1 - k_\alpha \mu_j^2)} = 1$$

where μ_j's are the positive zeros of the Legendre polynomial $P_{2n}(\mu)$ and k_α's are to be determined by solving the second equation above. Then, the emergent flux is given by

$$I_v(0, \mu) = F \sqrt{3}/4 \; H(\mu)$$

where F is the radiation flux, expresses the law of darkening.

The law of diffuse reflection from a medium characterized by the albedo ω_0 - which is the probability of scattering in an interaction - can be expressed as

$$I_v(0, \mu, \mu_i) = F/4\ \mu_i\ \omega_0\ /(\mu + \mu_i)\ H(\mu)\ H(\mu_i).$$

Fig.2 shows the variation of H(μ) with μ for some values of the parameter ω_0. Development of equations obeyed by the H-functions was most clearly obtained with the use of certain invariance principles, the complete set of which were first formulated by Prof. Chandrasekhar.

Principles of Invariance:

Radiation transfer theory discussed so far uses the local formulation as quantities defined locally alone are employed. In 1943 Ambarzumian formulated an invariance principle for the law of diffuse reflection:

- The law of diffuse reflection from an infinitely deep homogeneous plane parallel medium is invariant with respect to the addition (or subtraction) of layers of arbitrary finite optical thickness to (or from) the medium.

For the law of darkening, Prof. Chandrasekhar modified the principle as follows:

- The emergent distribution from a semi-infinite plane parallel medium is invariant to the addition (or subtraction) of layers of arbitrary optical thickness to (or from) the medium.

With the use of these principles, he showed that the H-function obeys the non-linear integral equation :

$$H(\mu) = 1 + \frac{\omega_0}{2} \mu\ H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu'$$

The above equation is for isotropic scattering, its generalized form for an arbitrary scattering law was also developed. The representation of the H-function given earlier can be readily derived from this equation using the Wick-Chandrasekhar method.

The invariance principles are not limited to semi-infinite media. The generalization of the principles to finite media made possible the solution of a large class of problems long considered impossible to solve. The four principles of Prof. Chandrasekhar can be stated as follows:

- Let the incident radiation flux F be in the forward direction μ_i (see Fig.3 a). Then the reflected intensity $I_\nu(\tau, \mu_r)$, at an (optical) distance τ in the reflected direction μ_r, results from the reflection of the attenuated incident flux F $\exp(-\tau/\mu_i)$ and the flux $I_\nu(\tau, \mu_f)$, in the forward direction μ_f, by the medium of thickness τ_1-τ.
- The intensity $I_\nu(\tau, \mu_f)$, in the forward direction at distance τ, (see Fig.3 b) results from the transmission of the incident flux through the thickness τ, and the reflection by the same surface of the intensity $I_\nu(\tau, \mu_r)$ in the reflected direction μ_r, which arises due to the presence of the medium of thickness τ_1-τ.
- The law of diffuse reflection $S_\nu(\tau_1, \mu_r, \mu_i)$ of a medium of thickness τ_1 is equivalent to reflection of part of the medium of thickness τ (part A in Fig.3 c) and the transmission by the same part of $I_\nu(\tau, \mu_r)$, which arises due to the presence of the medium of thickness τ_1-τ (part B).
- Just like the law of reflection, a law of transmission can also be introduced. The transmission law $T_\nu(\tau_1, \mu_f, \mu_0)$ of a medium of thickness τ_1 (see Fig.3 d) is equivalent to the transmission of thickness τ_1-τ (part B) of the attenuated incident flux F $\exp(-\tau/\mu_i)$ and $I_\nu(\tau, \mu_f)$, which is contributed by the part of thickness τ (part A).

These four invariance principles suffice to determine the radiation intensity inside the medium.

Chandrasekhar's X- and Y- functions:

Analysis of the four invariance principles shows that laws of diffuse reflection and transmission of a medium of optical thickness τ_1, viz., $S_\nu(\tau_1, \mu_r, \mu_i)$ and $T_\nu(\tau_1, \mu_f, \mu_i)$, can be expressed in terms of the solutions of two non-linear integral equations. These equations, in fact, result from a generalization of the equation for the H-functions. For the case of isotropic scattering, the X- and Y-functions obey the equations :

$$X(\mu) = 1 + \frac{\omega_0}{2}\mu \int_0^1 [X(\mu)\, X(\mu') - Y(\mu) Y(\mu')] \frac{d\mu'}{\mu + \mu'}$$

$$Y(\mu) = e^{-\tau_1/\mu} + \frac{\omega_0}{2}\mu \int_0^1 [Y(\mu)\, X(\mu') - X(\mu) Y(\mu')] \frac{d\mu'}{\mu + \mu'}$$

Solutions to problems in radiation transfer theory in finite plane parallel media can be expressed in terms of these functions.

A particularly interesting application, which is of interest in studies of structure of materials by scattering of radiation like neutrons, x-rays or light, in an inverse problem to determine the scattering law of a medium from the measured reflection and transmission laws. The incident radiation on the medium is usually at normal incidence. The scattering law - introduced as $p(\mu_0)$ earlier - contains information on the structure of the scattering sub-units. This method of using radiative transfer theory will be valid when multiple scattering is incoherent as is the case usually.

Transfer of polarized radiation:

The polarization state of light (as well as of neutrons) changes on scattering. According to Rayleigh's law, unpolarised light beam gets partially polarized on scattering from a particle. The ratio of intensities in directions parallel and perpendicular to the plane of scattering (which is made of the incident and reflected light) is in the ratio $\cos^2(\theta)$:1 where θ is the scattering angle. Thus light in the atmosphere is partially polarized, and the determination of the state of polarization - at different space points and directions of motion - is a very involved problem. The radiation field has to be characterized by specifying the intensity, degree of polarization, plane of polarization and ellipticity of polarization. This is usually done in terms of Stokes parameters: i) the intensity $I = I_l + I_r$, ii) the difference $Q = I_l - I_r$ (I_l and I_r are the intensities in two mutually perpendicular directions in a plane transverse to the direction of propagation), (iii) U and (iv) V, which specify the plane of polarization and ellipticity. Thus if the angles (θ, ϕ) represent the direction of motion of light in the atmosphere, then the four component vector $[I_l(\theta, \phi), I_r(\theta, \phi), U(\theta, \phi), V(\theta, \phi)]$ determines the polarization state of light at a spatial point. For unpolarised light, $I_l = I_r$ and $U = V = 0$. Radiation transfer equations in terms of the four component vector were first formulated and solved by Prof. Chandrasekhar.

Summary:

What is discussed above is, perhaps, the absolute minimum of the momentous contributions of Prof. Chandrasekhar to radiation transfer theory. As mentioned earlier, the subject is of immense importance today, particularly to those pursuing fission and fusion technologies. In the former, most of the aspects of chain reacting systems are determined by the neutron and γ-radiation fields. In all the fusion concepts, high temperatures, production, absorption and transfer of thermal radiation is of utmost importance. This is in addition to the transport of high energy neutrons and charged particles produced in these systems.

Finally, it is necessary to mention the related topic of rarefied gas dynamics [4]. As the rarefied gas is very dilute and sparse, its hydrodynamic description turns out to be inadequate. However, collisions between the molecules, and of the molecules with the container walls are important. Thus a linearized Boltzmann equation is used to describe the distribution function of the molecules in the gas. The mathematical structure of the resulting equations have a lot in common with those in radiation transfer theory. The methods and techniques originated by Prof. Chandrasekhar in radiation transfer theory have been of great use in all these fields, and they will continue to be used for years to come.

***Acknowledgments*:** I thank Dr. D. C. Sahni for his remarks and suggestions on the manuscript.

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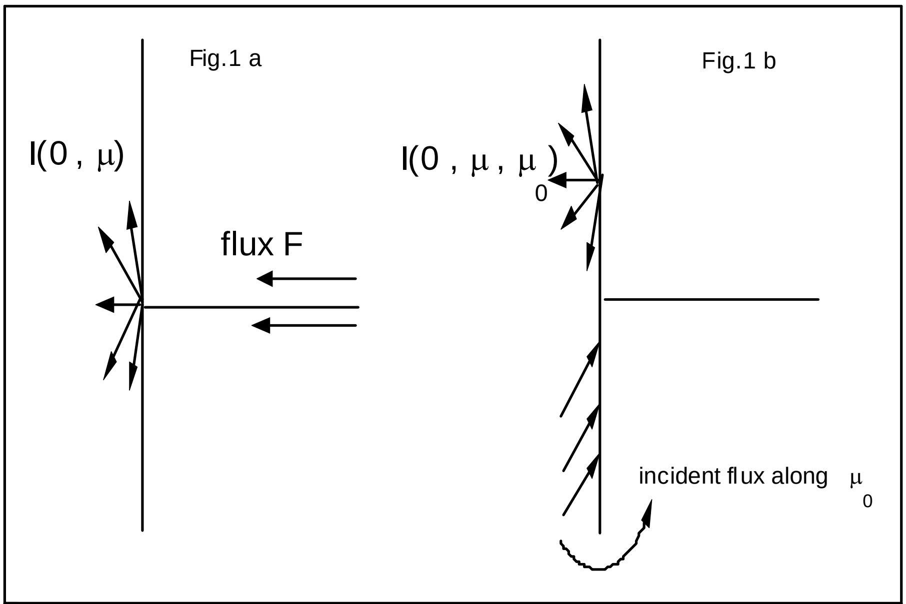
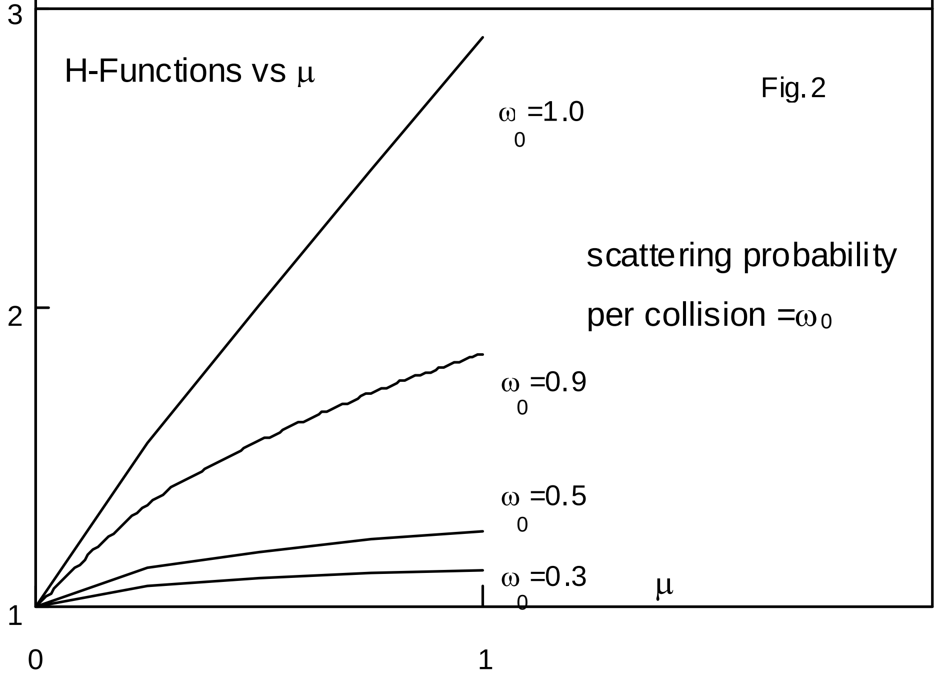


Fig.1 a
I(0 , μ)
flux F
Fig.1 b
I(0 , μ , μ0)
incident flux along μ0


H-Functions vs μ
Fig.2
ω0 =1.0
scattering probability
per collision =ω0
ω0 =0.9
ω0 =0.5
ω0 =0.3
μ
3
2
1
0
1

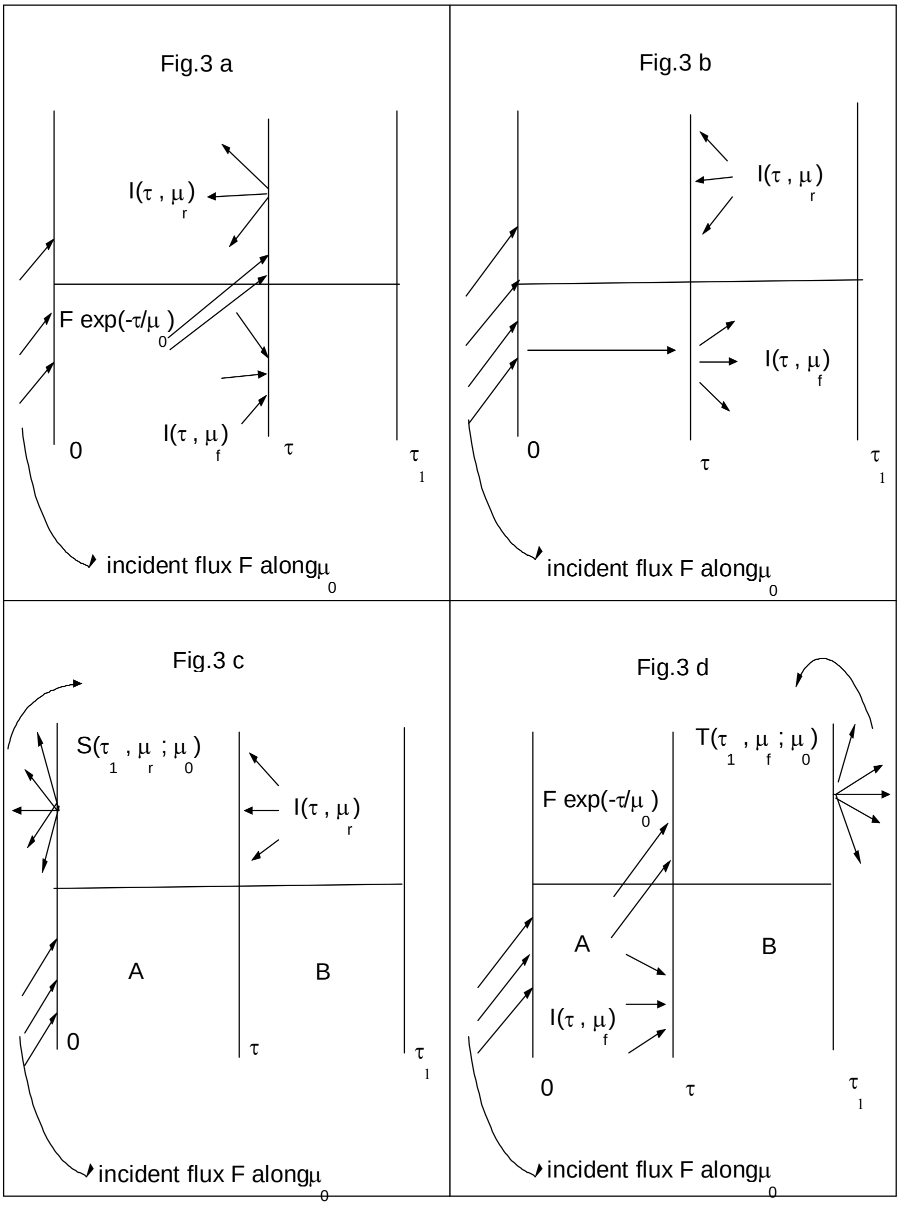


Fig.3 a
I(τ , μ_r)
F exp(-τ/μ_0)
I(τ , μ_f)
0
τ
τ_1
incident flux F along μ_0
Fig.3 b
I(τ , μ_r)
I(τ , μ_f)
0
τ
τ_1
incident flux F along μ_0
Fig.3 c
S(τ_1 , μ_r ; μ_0)
I(τ , μ_r)
A
B
0
τ
τ_1
incident flux F along μ_0
Fig.3 d
T(τ_1 , μ_f ; μ_0)
F exp(-τ/μ_0)
A
B
I(τ , μ_f)
0
τ
τ_1
incident flux F along μ_0