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S.V.G. MENON

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About the book

In nature there are several phenomena like thermal phase transitions or percolation processes which involve a multitude of length scales and / or time scales. For describing such phenomena, Kenneth Wilson, around 1970, put forward the renormalization group theory. The basic ideas and techniques of the theory are elaborated in this monograph using some simple models of ferromagnetic critical behavior. Brief outlines of applications to some of the related areas are also given. This monograph would provide a self contained introduction to beginners.

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I dedicate this book to the loving memory of my father.

Contents

Foreword

The subject of phase transitions has been at the frontier of condensed matter physics research for several decades. Continuous phase transitions are somewhat unique as several systems, for instance, a magnet near the Curie temperature, a fluid at the onset of condensation, etc., which are very distinct at a microscopic level, show a lot of similarities in their behaviour near the transition points. A physical theory to explain the universality observed in these systems, and capable of quantitative predictions, was lacking for a long period of time. Around 1971, Kenneth Wilson showed that the renormalization group theory is an adequate framework for describing phase transition phenomena. Since then, there has been a surge of activity in this field and many other related areas. Cooperative interaction among the constituent units is the key issue in all these fields.

Books on such specialized topics are indeed necessary for introducing the student community to newer areas of physics. The books published from the west are often out of reach of an Indian graduate student. To alleviate matters, the Indian Physics Association has been bringing out monographs on topics of current interest in physics. In the present volume by S. V. G. Menon, the renormalization group theory is introduced, and its basic concepts and techniques are elaborated, with applications to some of the key problems. Almost all the aspects are developed from a basic level and so it is quite self contained. Each chapter is accompanied by a list of important references which would be helpful for further study. I hope that graduate students and researchers desirous of learning renormalization group theory would find this monograph valuable.

R. Chidambaram
Chairman, Atomic Energy Commission

Preface

Renormalization group theory is a framework for describing those phenomena that involve a multitude of scales of variations of microscopic quantities. Systems in the vicinity of continuous phase transitions have spatial correlations at all length scales. There are other problems in percolation theory, polymer physics, etc., where the crux of the matter is again the occurrence of multiple length scales in spatial structures. Time scales and length scales of different sizes are involved in the dynamical evolution of systems close to phase transitions. Same is the situation in turbulence in fluid flow. These phenomena lacked a proper description till the advent of the renormalization group theory. Successful theories of physics till then could incorporate at best a few scales of variations of microscopic quantities. The basic aspects of the renormalization group theory were put forward, by Kenneth Wilson, in reference to the problem of continuous phase transitions. Since then, the theory has been developed further, and applied to a variety of problems in diverse fields involving cooperative behaviour. Now it is an accepted fact that the renormalization group theory is the main tool to be used in elucidating the finer aspects of many body physics. There is even the possibility of deriving statistical mechanics itself from very basic principles.

The renormalization group theory and the pertinent background material are introduced and applied to some important problems in this monograph. It grew out of a course of lectures I have given in the Theoretical Physics Division of the Bhabha Atomic Research Centre. Though the course was modeled along the works of S. Ma (1976) and M. E. Fisher (1982), my attempt was to make it simpler for a beginner by supplementing appropriate intermediate material from several other references cited later. The monograph begins with a historical survey of thermal phase transitions. The background material leading to the renormalization group theory is covered in the first three chapters. Then, the basic techniques of the theory are introduced and applied to magnetic critical phenomena in the next four chapters. The momentum space approach as well as the real space techniques are, thus, discussed in detail. Finally, brief outlines of applications of the theory to some of the related areas are presented in the last chapter. I have tried to discuss all the relevant aspects from a basic level so that the monograph can be read without any prior knowledge of the subject. Some elementary knowledge of statistical mechanics is expected, however, nothing more than the concepts of Gibb's canonical distribution, statistical averages and free energy are necessary. Sometimes the mathematical equations are lengthy, but there are no intrinsic complications and one can easily go through them. I hope that this monograph would provide a simple introduction to a fascinating field of modern theoretical physics.

I thank the executive committee (for the period 1991-1993) of the Indian Physics Association (IPA) for sponsoring the publication of this monograph. Dr. R. Chidambaram, Secretary to the Government of India, Department of Atomic Energy, and Chairman, Atomic Energy Commission, and also President of IPA, has been very generous to write a foreword to this monograph. Prof. R. Ramachandran, Director, The Institute of Mathematical Sciences, Madras, and Vice President of IPA, got the manuscript reviewed and gave the final approval for IPA's sponsorship. The reviewers had made several suggestions to improve and enlarge the original manuscript and I believe that the final form is much better. Dr. S. K. Gupta, General Secretary, IPA, and Shri. A. N. Nakra, Treasurer, IPA, have helped me in numerous ways to bring out this monograph. Besides, my colleagues have always been very cooperative. My wife, Lathika, and daughter, Sapna, provided all the necessary support from the family. I am grateful to one and all.

June 1993.
Bombay

S. V. G. Menon.

Historical Survey

The topic of phase transitions and critical phenomena has a history of more than one hundred years and what follows is a brief survey. The subject originated around 1869 with Thomas Andrews' experiments on carbon dioxide. During the first period, very important experiments on magnetic materials and binary alloys evolved along with theoretical developments which are known today as mean field theories. In 1944, Onsager published the exact statistical mechanical solution of a two dimensional magnetic system and initiated the second period in the historical development of the subject. During the period up to about 1965, similar exact and numerical calculations on model systems established the inadequacy of mean field theories in the neighborhood of phase transition points. Then, up to about 1971, several people attempted to put together the results of rigorous calculations and mean field theories. All this work culminated in formulating, empirically, the hypothesis of universality in the behaviour of systems near phase transition point. In 1971, K.G.Wilson developed the renormalization group approach as a new method for studying critical phenomena and thus laid a theoretical basis for understanding universal behaviour. Elaboration of this theory is the primary motif of this monograph.

Early Stage (1869 - 1944)

Even though it was known earlier that certain substances ceased to exist in the liquid phase above a certain temperature, it was Andrews' accurate measurements of the isotherms of carbon dioxide which established the continuity of the gaseous and liquid phases of matter. He introduced the term 'critical point' for a specific point (P_c, T_c, V_c) in the phase diagram at which the liquid and gaseous phases merged into a single fluid phase. He also showed that by a proper choice of a path in the phase diagram, one can pass from the liquid to gaseous phase without encountering any discontinuity in density. Few years later (1873) Van der Waals developed a generalization of the equation of state for ideal gases to provide a theoretical explanation of Andrew's isotherms for carbon dioxide. He argued that the attractive forces between the molecules

in the gas give rise to an internal pressure, which decreases the pressure in a gas, and due to the finite size of the molecules, the available volume for molecular motion is less than that of the container. Employing these ideas and invoking kinetic theory concepts to compute the internal pressure, Van der Waals proposed his famous equation of state. The modified equation of state explained the isotherms of carbon dioxide and also showed the existence of the critical point. Further, it brought out the idea of a universal equation of state in the sense that the isotherms of all gases merged into a single one when expressed in terms of reduced variables. Maxwell noted that below the critical temperature, Van der Waals equation showed a range of thermodynamically unstable densities and rectified this drawback with his now well known method of 'equal area construction'.

The observation of the striking phenomena associated with changes in the scattering of light by fluids near the critical temperature opened up the topic of density fluctuations. Fluids, which are transparent normally, show significant changes in color and finally become opaque as the critical temperature is approached. This phenomenon is known as critical opalescence and was explained by Smoluchowski (1908) and Einstein (1910) as arising out of large density fluctuations. Using Einstein's thermodynamic formula for the mean square density fluctuation

$$< (\triangle\rho) >^2 \;\; = \;\; \frac{\rho^2}{V} k_B T K_T,$$

where K_T is the isothermal compressibility, and Rayleigh's formula for incoherent light scattering cross-section, they had concluded that the intensity of scattered light (of wavelength λ) varies as $I \;\; \sim \;\; K_T/\lambda^4$. As the critical temperature is approached, K_T increases significantly and hence light scattering becomes quite predominant making the fluid opaque. This argument had not accounted for the presence of correlation between density fluctuations at different space points in the fluid. To overcome this drawback, Ornstein and Zernike (in 1914) introduced the concept of the density-density correlation function or the pair distribution function $g(r)$. They derived a relation connecting K_T and the volume integral of $g(r)$ which showed that large spatial correlations are developed in a fluid as the critical temperature is approached. Introducing another correlation function, which signifies the direct interaction between two atoms separated in space, they derived an integral equation for $g(r)$ which yielded the general result

$$g(r) - 1 \;\; \sim \;\; \frac{1}{r} \exp(-\frac{r}{\xi}).$$

The parameter ξ, which was shown to be proportional to $\sqrt{K_T}$, is called the correlation length and it characterizes the spatial length scale over which

correlations exit in a fluid. Now, as critical temperature is approached, K_T increases and hence large spatial correlations develop. Appropriately, the assumption of incoherent scattering had to be replaced with that of coherent scattering from the correlated regions. The scattered light intensity corresponding to a wave vector change q is then found to be

$$I(q) \sim \frac{K_T \lambda^{-4}}{\xi^{-2} + q^2}.$$

Since $q \sim \lambda^{-1}$, at the critical point the wavelength dependence of $I(q)$ is λ^{-2} in comparison to λ^{-4} derived earlier.

Curie in 1895 made detailed investigations on the temperature dependence of magnetic properties of materials and put forward ideas showing similarities to properties of fluids. Earlier (1889), Hopkinson had introduced the term critical temperature (T_c) above which the materials lost the magnetic properties abruptly. Taking pressure and specific volume analogous to magnetic field and magnetization, Curie's analogy led to similarities between the gaseous phase and paramagnetic state at temperatures above T_c and the ferromagnetic phase and liquid phase below T_c . Molecules of a magnetic material were themselves modeled as tiny magnets and Langevin, employing statistical mechanics, derived an equation of state (relating magnetization (m), applied field (h) and temperature (T)) which was analogous to the ideal gas equation for fluids. This theory, known as Langevin's theory of paramagnetism, had explained the relation $\chi = c/T$ between the magnetic susceptibility and temperature derived experimentally by Curie. The magnet-fluid analogy led Weiss in 1907 to postulate an internal field similar to the internal pressure introduced by Van der Waals for fluids. The internal field was to represent the effect of interaction between a molecule with other molecules in the material. On incorporating the internal field into Langevin's magnetic equation of state, Weiss found the existence of a critical temperature (known as Curie temperature). Above the critical temperature, the material behaved as a paramagnet while for lower temperature it acquired non-zero magnetization. The modified equation of state also led to the now well known Curie-Weiss law $\chi = c/(T - T_c)$ for the susceptibility.

With the use of x-ray diffraction techniques, ordered arrangement of atoms in binary alloys such as that of Cu and Au was established in the early 1920s. On increasing the temperature, destruction of atomic order accompanied by an anomalous increase in specific heat of alloys was observed. In 1934, Bragg and Williams introduced the concept of an order parameter s to characterize the degree of atomic order in the alloys. Their statistical mechanical calculation for the temperature dependence of s, along the lines of Weiss theory of ferromagnetism, led to an 'equation of state' which showed that s decreased

continuously and approached zero at a critical temperature T_c . The work of Bragg and Williams brought out the fact that short range forces between atoms can compound together in a cooperative manner to establish long ranged correlations.

Magnetization of a ferromagnetic material at zero field, density difference between the gaseous and liquid phases at P_c and the order parameter in the case of binary alloys vary in a continuous manner with respect to temperature across T_c, being zero above T_c and non-zero below T_c . In the current terminology, these transitions are called continuous phase transitions. Thus these transitions are qualitatively different from those involving a discontinuous change in density in a fluid at a pressure different from P_c. A discontinuous change in density implies a discontinuity in the first derivative of Gibb's free energy and hence discontinuous transitions are also said to be of first order. Continuous transitions are accompanied by a divergence of specific heat across T_c . Thus these transitions with a continuous first derivative of free energy but a divergent second derivative are commonly known as second order phase transitions.

During the early period, simple models retaining the essential aspects of a many body system, which cooperatively interact near T_c, were proposed for quantitative study of phase transitions. In 1925, Lenz suggested to his student Ising a model consisting of classical spin variables (representing magnetic moments of atoms) at the sites of a lattice to represent a magnetic material. Every spin variable can point up or down and interact with its nearest neighbors such that two parallel spins have a lower energy state in comparison to two antiparallel spins. Ising solved the statistical mechanical problem in one dimension, but found no phase transition. Thereafter, the model came to be known as Ising model. Somewhat later (around 1936) Peierls gave arguments to show that in two dimension, the model predicted a non-zero magnetization at a finite temperature. A more realistic model wherein a three dimensional spin vector occupied the sites of a lattice was proposed by Heisenberg in 1928. These and other related models will be introduced in the first chapter.

Around 1937, Landau unified the theories of continuous phase transitions. He generalized the concept of order parameter, introduced by Bragg and Williams for binary alloys, to characterize all continuous phase transitions. The density difference between the liquid and gaseous phases and magnetization are the order parameters for the gas-liquid and magnetic transitions. In the vicinity of T_c , Landau developed a Taylor expansion for Gibbs free energy in terms of the order parameter. Employing changes in the symmetry of the system across the transition temperature, the temperature dependence of the coefficients in the free energy expansion was parameterized and then minimization of free energy was shown to yield the appearance of order below

T_c . Landau's theory also brought to focus the importance of critical exponents to characterize the divergence of thermodynamic quantities across T_c . Within this theory, all continuous phase transitions are found to have the same critical exponents. These aspects will be discussed in more detail in the second chapter.

Middle Period (1944 - 1971)

The inadequacy of all the theoretical work on phase transitions in the early period was exposed by the work of Onsager in 1944. He analytically solved the two dimensional Ising model in zero external field and showed that at a critical temperature T_c, the magnetic contribution to specific heat diverged logarithmically. Mean field theories of Landau's type predicted only a discontinuity in the specific heat for all spatial dimensions. The free energy was shown to be non-analytic at T_c , thus making Landau's expansion invalid. The divergence of correlation length at T_c was found to be at variance with the predictions of Ornstein-Zernike theory. Later (1949), Onsager also showed that the temperature dependence of magnetization below T_c was different from that predicted by mean field theories. The three dimensional Ising model is not yet solved analytically. However, perturbation series expansions at high and low temperatures together with asymptotic expansions employing Pade approximations developed by Domb and others clearly established the inadequacy of mean field theories. It should be mentioned here that somewhat later in the 70s, Kac, Uhlenbeck and others established the correctness of the mean field theory results for systems interacting via very long range forces. The mean field results did not accord with experiments since inter-molecular forces are generally short ranged. From the results of calculations for different type of lattices (cubic, face centered cubic, etc.) and accumulated experimental data, it became clear that critical behaviour strongly depended on the spatial dimension and the dimension of the spin variable but was insensitive to the details of the system at small length scales.

In the 1960s, several workers (Widom, Domb and Hunter, Patashinskii and Pokrovskii, Fisher, Griffiths) attempted to incorporate the experimental results and 'exact results' on models into the mean field theory expressions. This development led to what is today known as scaling theories which hypothesized certain expressions for free energy and correlation functions. These expressions which involved scaled variables like $m/(T - T_c)^\beta$ were analogous to those predicted by mean field theories, however, the functional forms and exponents (numbers like β) were determined from known results. Scaling hypotheses predicted relations between exponents and certain universal functional forms similar to the universal equation of state suggested by the Van

der Waals equation. Tests of the predictions with exact and experimental results indicated different universality classes, each class depending on the spatial dimension and dimension of the spin variable. Kadanoff in an important paper (1966) attempted a theoretical justification of scaling hypotheses. He observed that near the critical temperature, the basic length scale of the system is the correlation length which is much larger than other length scales like inter-particle spacing, interaction range etc. He argued that from a given theoretical description of the system, it is possible to construct an equivalent coarse grained description such that the coarse graining scale is much smaller than the correlation length. Employing the equivalence of the two descriptions and invoking certain other assumptions, Kadanoff derived the scaling hypotheses. Development of scaling theories and Kadanoff's ideas are discussed in the third chapter.

Renormalization Group Period (After 1971)

Physical phenomena observed in nature are characterized by a great diversity of length scales. Matter at the molecular level shows features at a scale of the order of 10^{-8} centimeter, but at the macroscopic level the scales involved can vary from a fraction of a centimeter to several thousand kilometers. Almost all theories of physics exploit the fact that for describing a class of phenomena characterized by length scales in a certain range, details of the system within the range can be ignored or suitably approximated. Near the phase transition point, due to the co-operative interaction of molecules, chunks of matter of all sizes exist and a fundamental theory should incorporate this aspect. K.G.Wilson showed that the renormalization group theory, originally developed in the 1950s in connection with field theories of elementary particles, is an appropriate framework for understanding the universality and also for detailed calculations in phase transition theory. He translated Kadanoff's coarse graining concept to the wave vector (or momentum) space and developed the idea, that two descriptions differing in the basic length scales (but both smaller than the correlation length) are equivalent, into a symmetry principle for critical phenomena. Thus by repeated application of coarse graining of the system, features at successively larger length scales could be incorporated in the formalism. Then, rather general considerations led to the derivation of scaling hypotheses proposed in an adhoc manner earlier. Using the Landau-Ginzburg model hamiltonian (explained in the second chapter) he also obtained (approximate) values for critical exponents in good agreement with experiments. Together with M.E.Fisher, Wilson also pioneered a perturbation scheme (known as ϵ - expansion) in the parameter $\epsilon = 4 - d$ where d is the spatial dimension of the system and showed that mean field results are

exact for $d \geq 4$. All these developments are discussed systematically in chapters four to six. The usefulness of Kadanoff's coarse graining concept in real space to study discrete spin models was shown by Th.Niemeyer and J.M.J. van Leeuwen (1974), Kadanoff(1975) and others. These methods known as real space renormalization techniques are briefly introduced in chapter seven.

Like matter near the phase transition point, there are other systems where length scales of all sizes are important. Percolation of fluids through a solid matrix, turbulence in fluids, size and shape of polymer chains in solutions, diffusion of particles through random structures, chaotic maps, etc. are some examples. Renormalization group theory has been applied in these areas with significant success during the last decade. A few of these applications are discussed in the last chapter.

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Chapter 1

Basic Aspects

This chapter is devoted to the basic aspects of second order critical phenomena. The important experimental facts are summarized first. Then, the mathematical models are introduced, and the statistical theory is outlined.

1.1 Critical Phenomena

A whole lot of physics deals with the behaviour of macroscopic systems when external conditions are varied. The basic aim of a theory is to provide an understanding of the behaviour, to classify the systems based on their behaviour, and to explain the unifying features of their behaviour if any. A macroscopic theory deals with quantities like mass density, energy density, magnetization, current density, etc., which are refered to as mechanical variables. There are also quantities like applied temperature, pressure, electric field, magnetic field, etc., which are called applied fields. These fields characterize the environment or reservoir with which the system is in contact. In most of the phenomena, the mechanical variables are uniquely fixed by the applied fields.

There are certain cases where a mechanical variable is not completely fixed by the applied fields. For example, at 100^oC and atmospheric pressure, the density of H_2O has two values, one corresponding to the vapor phase and the other corresponding to the liquid phase. In fact this is true for all points on the curve in the P-T diagram (Figure 1.1) which terminates at (P_c, T_c) known as the liquid-gas critical point. Another example is the ferromagnetic phase of materials like Fe, Co, Ni, etc. In this case the magnetization vector $\mathbf{m}$ is not fixed when the applied field $h = 0$. The paramagnetic phase where $\mathbf{m} = 0$ for $h = 0$ prevails for $T > T_c$, a critical value. The point $(0, T_c)$, in Figure 1.2, known as the magnetic critical point is similar to the liquid-gas critical point.

Phenomena observed near critical points are called critical phenomena. Mechanical variables like density (ρ) and magnetization ($\mathbf{m}$) which are not uniquely fixed by the applied fields are called order parameters. There are

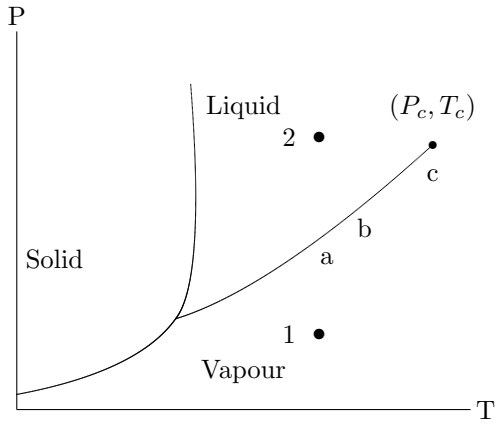

Figure 1.1: P - T Phase Diagram for Gas-Liquid Transition.

other examples like the superfluid and superconducting critical points. In these cases a macroscopic theory employs quantum amplitudes, which are generally complex, as order parameters. Experimental observations show that many of the critical phenomena have several common features.

1.1.1 Ferromagnetism

The source of magnetic moments of atoms of ferromagnetic materials is the spin of electrons in incomplete atomic shells. For transition metals (Fe, Co, Ni) the d and f shells are incomplete. The spins of electrons (in different atoms) have a lower energy when they are parallel and the basic reason is the quantum mechanical exchange effect. The crystal structural features sometimes make all spins to be restricted to a certain crystal axis or to a crystal plane. Thus there are uniaxial or planar ferromagnets in addition to isotropic ferromagnets. At $T = 0$, all spins are in the same direction even though the direction is arbitrary. As T is increased, thermal agitation randomizes the spin direction, but still a large fraction of the spins is in the same direction for long time intervals. For $T > T_c$, the critical temperature, the net number of spins in any direction is zero. However, for T slightly above T_c, there are large spin patches (in comparison to lattice spacing) where alignment is achieved. For T slightly below T_c, there are spin patches of macroscopic sizes as well. It takes a long time for the short range exchange effect to turn around large spin patches which exist near T_c. Thus relaxation near T_c is very slow. This and similar other experimental observations point to the fact that many of the features of ferromagnetism are due to the presence of large spin patches near T_c.

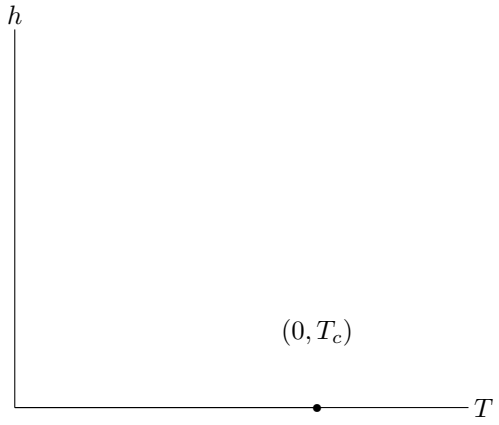

Figure 1.2: h - T Phase Diagram for Magnetic Transition.

1.1.2 Exponent β

When $h = 0$, magnetization is a decreasing function of T for $T < T_c$. The two curves shown in Figure 1.3 are the two possible (non-unique) values of $\mathbf{m}$ obtained for a uniaxial ferromagnet. It is natural to ask about the nature of this function and the simplest function having the required shape is a parabola,

$$m^2 \sim T_c - T.$$

Then the temperature dependence of $\mathbf{m}$ is

$$|\mathbf{m}| \sim (T_c - T)^{1/2}.$$

However, the observed nature of the curve is not a parabola. In fact one finds that

$$|\mathbf{m}| \ \sim \ (T_c - T)^{\beta},$$

with $\beta \approx 0.35$ and $T_c \sim 69.3^o k$ for $YFeO_3$. Surprisingly, the same value of β is found for many systems as if it is a universal number. Values of β for some materials are given in Table 1.1. β is one of the several critical exponents introduced below.

1.1.3 Liquid-Gas Critical Point

Historically, the first critical point to be discovered was in carbon dioxide. Consider a sealed tube containing CO_2 at an overall density of $0.5 gm/cc$ at $T = 29^oC$. The corresponding pressure is 72 atmospheres. At this point, shown as point a in Figure 1.1, there is clearly liquid and vapor. If T is raised to 30^oC (point b in the figure), the density of liquid and vapor comes closer.

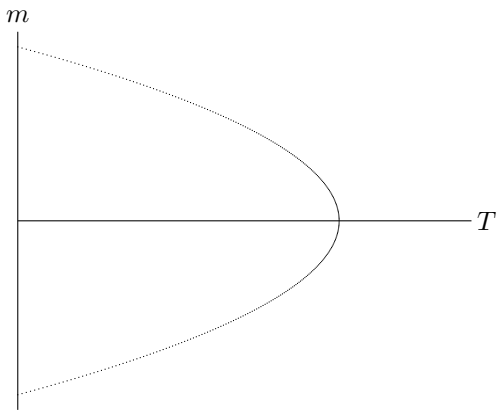

Figure 1.3: Magnetization - Temperature Curves.

Table 1.1: Exponent β

$Material$	$T(^{o}k)$	β
Fe	1044.0	0.34 ± 0.02
Ni	631.6	0.33 ± 0.03
$YFeO_3$	69.3	0.35 ± 0.02

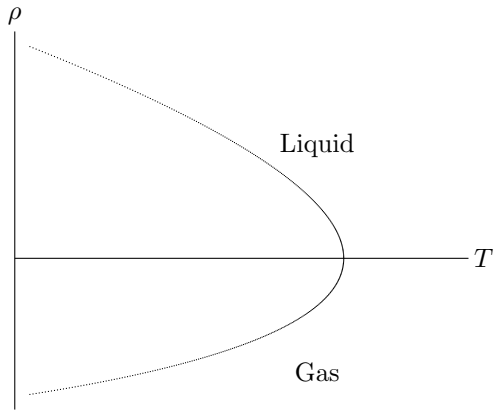

Figure 1.4: Density - Temperature Curves.

At 31^oC, one observes the phenomenon of critical opalescence. When CO_2, which is transparent to visible light, is illuminated, an orange tinge is found if viewed from the forward direction and a bluish tinge is observed from a normal direction. If T is raised by a small amount (point c in the figure), opalescence disappears and the two phases, vapor and liquid, also disappear leaving behind a homogeneous fluid. One can go from point-1 to point-2, in the phase diagram (of Figure 1.1), along a path of continuous density change or along a path with a discontinuous density change as the vapor pressure curve is crossed. As one approaches the critical point along the vapor pressure curve, liquid density (ρ_l) and vapor density (ρ_v) come closer as shown in Figure 1.4. Again, the dependence of $\rho_l - \rho_v$ on $T_c - T$ is found to be

$$\rho_l - \rho_v \ \sim \ (T_c - T)^{\beta}.$$

Very accurate measurements show that $\beta \sim 0.32$. More importantly, β is found to be independent of the type of fluid. The same value (within experimental error) is found for H_2O, liquid metals, He_3, He_4, Xe etc. Thus once again it appears that β is a universal number. Furthermore, the data on different liquids when expressed in terms of scaled variables

$$t_1 = \frac{\rho_l - \rho_v}{\rho_c}, \quad t_2 = \frac{T_c - T}{T_c}$$

are found to fall on a universal curve within experimental error.

1.1.4 Binary Mixture

Another system which has been investigated is a mixture of two chemical compounds A and B, which mix together at higher temperature but separate

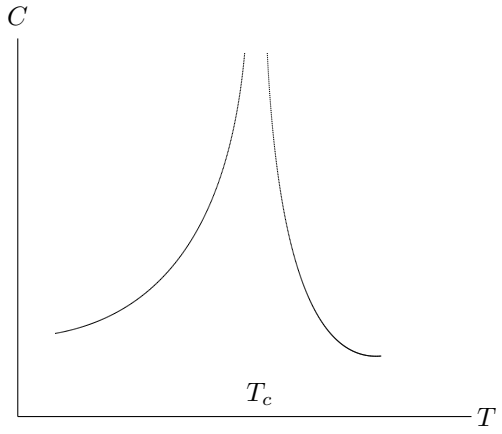

Figure 1.5: Specific Heat Vs Temperature.

into two phases at lower temperature. If N_1 and N_2 are the concentrations of compound A in the phases 1 and 2, near a critical value T_c, it is seen that

$$N_1 - N_2 \ \sim \ (T_c - T)^{\beta}.$$

For systems obtained by dissolving alkali metals (Na, Li, Ca) in NH_3, β has the same value quoted before.

1.1.5 Exponent α

The specific heat C of systems at the ferromagnetic critical point or the liquid-gas critical point is found to diverge as the critical temperature is approached. See Figure 1.5. The divergence is characterized in terms of critical exponents α and α' as

$$C \ \sim \ \begin{cases} (T - T_c)^{-\alpha}, & T > T_c \\ (T_c - T)^{-\alpha'}, & T < T_c. \end{cases}$$

For argon, α is found to be in the range of 1/8 to 1/9 and similar results are found for other fluids. Further, α is 0.12 ± 0.01 and 0.1 ± 0.03 for the magnetic materials Fe and Ni respectively.

1.1.6 Exponent γ

The zero field susceptibility (χ) of ferromagnetic materials diverges near T_c. Data show that χ behaves as

$$\chi \ \sim \ \begin{cases} (T - T_c)^{-\gamma} & T > T_c \\ (T_c - T)^{-\gamma'} & T < T_c. \end{cases} ,$$

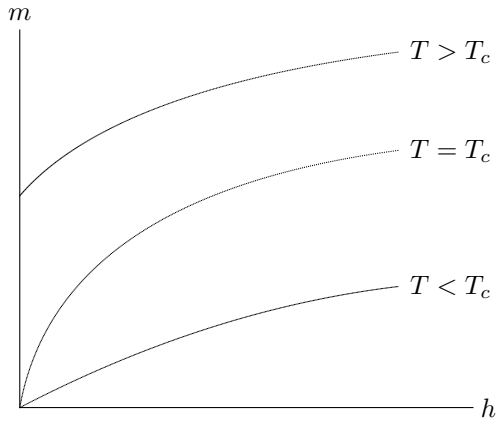

Figure 1.6: Magnetization Vs Applied Field.

and $\gamma = \gamma' \approx 1.33 \pm 0.1$ for Fe, Ni and Gd. χ measures the ease of magnetizing a material and it should diverge at T_c because of spontaneous magnetization. The analogous parameter for fluids is the isothermal compressibility

$$K_T = \frac{1}{\rho}(\frac{\partial \rho}{\partial p})_T,$$

and is found to become large near T_c. Typical values of γ defined as

$$K_T \sim \ (T - T_c)^{-\gamma},$$

are between 1.23 and 1.24 for several fluids.

1.1.7 Exponent δ

The variation of the order parameter on the critical isotherm is obtained by fixing $T = T_c$ and varying h (or P) in the magnetic (or liquid-gas) case. For small h one finds that

$$m \ \sim \ h^{1/\delta},$$

for $T = T_c$. See Figure 1.6. Values of the exponent δ range from 4.6 to 4.8 for uniaxial to isotropic ferromagnets. For fluids, δ is defined as

$$(\rho - \rho_c) \ \sim \ (P - P_c)^{1/\delta},$$

and δ varies from 4.2 to 4.8.

1.1.8 Definition of Exponents

When a function $f(x)$ behaves like x^λ for small x, it is written as

$$f(x) \ \sim \ x^\lambda \ as \ x \to 0.$$

Table 1.2: Exponents (Approximate)

n	α	γ	δ	β	ν
1	0.11	1.235	4.8	0.32	0.63
2	0.00	1.315	4.7	0.34	$--$
3	0.14	1.356	4.6	0.36	0.70

It means that

$$\lim_{x\to 0} \frac{\ln(f(x))}{\ln(x)} = \lambda.$$

This definition does not require the specification of the constant of proportionality as in $f(x) = Ax^{\lambda}$. Further, for $f(x) = \ln(x)$, $\lambda = 0$. It should be noted that $f(x)$ behaves like x^{λ} only for small values of x, in fact, the general form of $f(x)$ would be as

$$f(x) = Ax^{\lambda}\{1 + a_{-r}x^{-r} + \cdots + a_1 x + a_2 x^2 + \cdots\}.$$

1.1.9 Order Parameter Dimension

It was noted earlier that mechanical variables, which are not uniquely defined by specific values of applied fields, are generally called order parameters. For fluids the parameter of interest is $\rho_l - \rho_v$ while for fluid mixtures it is the difference in concentration $N_1 - N_2$. For superfluids, the parameter that characterizes the transition is a macroscopic wave function

$$\psi = \psi_1 + \imath\psi_2,$$

and hence has two components in comparison to the single component in earlier cases. For ferromagnets, the magnetization vector $\mathbf{m}$ is the order parameter. Uniaxial magnets with an easy axis of magnetization are described by a single component, $n = 1$, order parameter while planar and isotropic magnets require $n = 2$ and $n = 3$ respectively. Thus there are situations where the order parameter has many components. The various exponents introduced earlier are found to have a weak dependence on n. See Table 1.2.

1.1.10 Fluctuation of Order Parameter

In general, the order parameter is denoted by s. If the details of the spin arrangement in a ferromagnet are probed, it will be seen that the spin alignment varies in space and time. At a specific time, one can see a spin configuration.

It is useful to define a quantity called spin density $s(\mathbf{x})$ so that $s(\mathbf{x})d\mathbf{x}$ is the total spin in $d\mathbf{x}$ around the point $\mathbf{x}$. For simplicity, the case of a single component order parameter is considered here. Since thermal agitation is the main agency which disturbs the spin alignment, the spin configuration is decided by statistical laws. The net magnetization measured is the statistical average of $s(\mathbf{x})$, i.e. $m =< s(\mathbf{x}) >$. For every spin configuration, there is an associated energy E and the relative probability of occurrence of the configuration is given by the Boltzmann factor $\exp(-E/k_BT)$, where k_B is the Boltzmann constant. The spin configuration can be probed by scattering experiments using neutrons since neutrons have magnetic moments. The scattering cross-section γ_{fi} (associated with a momentum change from $\mathbf{p}_i$ to $\mathbf{p}_f$) depends on the local spin density. In the Born approximation one has

$$\gamma_{fi} \sim \Big\langle \Big| \int\limits_V \exp(-\imath \mathbf{p}_f \cdot \mathbf{x}) s(\mathbf{x}) \exp(\imath \mathbf{p}_i \cdot \mathbf{x}) d\mathbf{x} \Big|^2 \Big\rangle,$$

where $< \cdots >$ denotes averaging over various spin configurations with the corresponding probabilities. With periodic boundary conditions over the edges of the material of volume V, the Fourier modes

$$\phi_{\mathbf{k}} = \frac{1}{\sqrt{V}} \exp(\imath \mathbf{k} \cdot \mathbf{x}),$$

form a complete set of functions. Here, the wave vector component $k_i = 2\pi n/L$ where n is an integer and $V = L^3$. Then $s(\mathbf{x})$ can be expanded as

$$\begin{aligned} s(\mathbf{x}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \exp(\imath \mathbf{k} \cdot \mathbf{x}) s_{\mathbf{k}}, \\ s_{\mathbf{k}} &= \frac{1}{\sqrt{V}} \int\limits_V \exp(-\imath \mathbf{k} \cdot \mathbf{x}) s(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Using the orthogonality of the Fourier modes, one easily gets

$$\begin{aligned} \gamma_{fi} &\sim < |s_{\mathbf{k}}|^2 > V, \\ \mathbf{k} &= \mathbf{p}_f - \mathbf{p}_i. \end{aligned}$$

Scattering experiments show that γ_{fi} diverges (Figure 1.7) for $k \approx 0$ (i.e. for forward scattering) when $T = T_c$. The divergence can be expressed as

$$\gamma_{fi} \sim V \; k^{-2+\eta}.$$

and the exponent η takes values around 0.07. The scattering cross-section can be related to the Fourier transform of the correlation function of spin density. Now, $s(\mathbf{x}) - < s >$ is the deviation of spin density from its mean value and

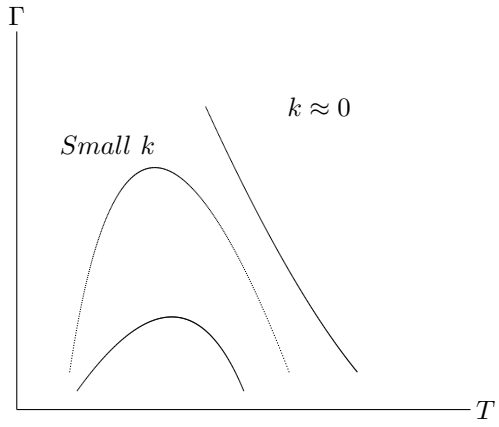

Figure 1.7: Scattering Cross-section Vs Temperature.

hence $< [s(\mathbf{x}) - < s >][s(0) - < s >] >$ is the spatial correlation function of spin density. Its Fourier transform $G(\mathbf{k})$ is

$$\begin{aligned} G(\mathbf{k}) &= \int_V d\mathbf{x} \exp(-\imath \mathbf{k} \cdot \mathbf{x}) < [s(\mathbf{x}) - < s >][s(0) - < s >] > \\ &= \int_V d(\mathbf{x} - \mathbf{x}_1) \exp\{-\imath \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_1)\} \Big\langle [s(\mathbf{x}) - < s >][s(\mathbf{x}_1) - < s >] \Big\rangle. \end{aligned}$$

The last step follows since any point $\mathbf{x}_1$ can be taken as the origin. Since the r.h.s is independent of $\mathbf{x}_1$, $G(\mathbf{k})$ can be written as

$$G(\mathbf{k}) = \frac{1}{V} \int_V d\mathbf{x}_1 \int_V d\mathbf{x} \exp\{-\imath \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_1)\} \Big\langle \Big[s(\mathbf{x}) - < s > \Big] \Big[(s(\mathbf{x}_1) - < s > \Big] \Big\rangle.$$

Now, note that the magnetization $m =< s(\mathbf{x}) >$ is independent of $\mathbf{x}$ for a homogeneous material. Then, substituting for $s(\mathbf{x})$ and $s(\mathbf{x}_1)$ in terms of Fourier modes, one easily finds

$$G(\mathbf{k}) =< |s_{\mathbf{k}}|^2 > .$$

Thus γ_{fi} measures the Fourier transform $G(\mathbf{k})$ of the correlation function. The divergence of γ_{fi} for small $\mathbf{k}$ shows that

$$G(0) = \int_V d\mathbf{x} \Big\langle [s(\mathbf{x}) - < s >][s(0) - < s >] \Big\rangle,$$

diverges as $T \to T_c$. Since $s(\mathbf{x})$ at all $\mathbf{x}$ are bounded quantities, the divergence should be attributed to the presence of very large regions where spins are correlated. Thus one is led to the conclusion that there are spin patches of large sizes in the system near T_c.

1.1.11 Correlation Length

The exchange interaction which aligns the spins is a short range interaction. Thermal agitation which randomizes the spin alignment is uncorrelated. Thus at high temperature, where the thermal agitation is more predominant, one expects the correlation function

$$G(\mathbf{x}) =< [s(\mathbf{x}) - < s >][s(0) - < s >] >,$$

to fall off rapidly. At T close to T_c, the presence of large spin patches indicates that spins at large distances are correlated. As will become evident later, rather general models show that the correlation function falls off as

$$G(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|} \exp(-|\mathbf{x}|/\xi),$$

for large values of $|\mathbf{x}|$ (in units of inter atomic spacing). The parameter ξ yields the typical length scale over which spins are correlated and is called the correlation length. For a crude picture, one may take ξ as the size of the largest spin patch. For $T \gg T_c$, ξ is of the order of few lattice spacing. The phenomenon of critical opalescence or diverging scattering cross-section shows that ξ diverges as T_c is approached. The variation of ξ w.r.t temperature can be described as

$$\xi \sim (T - T_c)^{-\nu},$$

where ν is a new exponent and its value ranges from 0.63 to 0.7 as n goes from 1 to 3. The divergence of ξ, or the presence of spin patches of large size near T_c, is the main clue which provides an understanding of critical phenomena.

1.2 Mathematical Models

In this section, some mathematical models used to study critical phenomena are introduced. The occurrence of large correlation length near the critical point shows that the problem of critical behaviour is a many body problem. Therefore, very simple models may be studied to obtain a conceptual understanding of the phenomena. The detailed quantum mechanical solution of a many body problem, even if possible, can not provide such an understanding. In the following sections, simple models with particular emphasis on the magnetic critical phenomena are discussed.

1.2.1 Ising model

In the Ising model, the details of atomic structure and crystal structure are ignored and one imagines space to be divided into cells of certain volume v

and each cell is represented by a lattice point. To each lattice point, a spin variable s_i, which can take values ± 1, is assigned and then an exchange type interaction between the spins is postulated. If i, j, etc. denote the lattice points, the total number being N, the hamiltonian of the system is

$$H(\{s_i\}) = -h \sum_i s_i - J \sum_{<i,j>} s_i s_j,$$

where h is the external field and J (> 0) is the exchange interaction parameter. The first term accounts for interaction of spins with the external field. The second term yields a negative contribution from a pair of parallel spins and a positive contribution from anti-parallel spins. The symbol $\sum_{<i,j>}$ indicates that summation is over nearest neighbour pairs. The total number of terms in the sum is $N\ z_{nn}$ where z_{nn} is the number of nearest neighbours ($= 2$ in $1-D$, 4 in $2-D$ and 6 in $3-D$). The lattice can be of several types, square or triangular in $2-D$, simple cubic in $3-D$, etc. Thus in a square lattice in d-dimension, the total number of spin variables are $N = L^d$ where L is the side length of the lattice in units of lattice spacing. The statistical mechanical properties of the model can be expressed in terms of the canonical partition function, which is defined as

$$Z_N(T, h) = \sum_{config} \exp\Big[-\frac{H}{k_B T}\Big].$$

The symbol $\sum_{config}$ indicates summation over all the 2^N spin configurations.

1.2.2 Lattice Gas Model

The lattice gas model is a simple model to characterize a fluid and can be formulated exactly like the Ising model. First of all, one assumes that the position of atoms in a fluid can be only at the sites of a lattice. A number n_i is assigned to the i^{th} lattice site, and it can take values 1 or 0 depending on whether the site is occupied or not. Thus, at most one atom can occupy a site. Generally, there is a repulsive interaction when two atoms approach very close to each other and this fact is modeled by the restriction that at most one atom can occupy a lattice site. Assuming a nearest neighbour (attractive) interaction energy $-\epsilon$, the total energy of a configuration having N' particles can be expressed as

$$H(N') = -\epsilon N'_p \quad = -\epsilon \sum_{<i,j>} n_i n_j,$$

where N'_p is the number of neighbour pairs of the configuration. The kinetic energy of the particles is not considered since it contributes only the ideal gas

terms (to the thermodynamic quantities), which are unimportant in discussing phase changes. The total number of particles in the configuration is

$$N' = \sum_i n_i.$$

The canonical partition function is then given by

$$Z_{N'} = \sum_{config\ -\ N'} \exp\Big[-\frac{H(N')}{k_BT}\Big],$$

where $\sum_{config\ -\ N'}$ represents summation over all the distinct configurations of N' particles on N lattice points. Note that this number is $N!/(N-N')!N'!$. The lattice gas model can be made identical to the Ising model by considering the grand partition function for N particles,

$$\begin{aligned} Z_G &= \sum_{N'} Z_{N'} \exp\Big[\frac{\mu N'}{k_BT}\Big] \\ &= \sum_{N'} \sum_{config\ -\ N'} \exp\Big[-\frac{H(N') - \mu N')}{k_BT}\Big], \end{aligned}$$

where μ is the chemical potential. Substituting for $H(N')$ and N' and observing that

$$\sum_{N'} \frac{N!}{(N-N')!N'!} = 2^N,$$

Z_G can be written as

$$Z_G = \sum_{config} \exp\Big[-\frac{H_e}{k_BT}\Big].$$

Here, the effective hamiltonian H_e is defined as

$$H_e = -\epsilon \sum_{<i,j>} n_i n_j - \mu \sum_i n_i,$$

and $\sum_{config}$ indicates summation over all the 2^N possible sets of $\{n_i\}$ values. Thus the calculation of Z_G for the lattice gas model is identical to that of Z_N for the Ising model. For establishing exact equivalence, a spin variable s_i can be introduced as

$$n_i = \frac{1}{2}(s_i + 1).$$

Then H_e can be written as a Ising hamiltonian. The coupling constant and the 'field strength' of the equivalent hamiltonian are

$$\begin{aligned} J_{eff} &= \frac{\epsilon}{4} \\ h_{eff} &= \frac{\epsilon}{4} z_{nn} - \frac{\mu}{2}. \end{aligned}$$

The lattice gas model can also be used to describe the binary mixture. A usual convention is that the i^{th} site is occupied by an A-atom if $n_i = 1$ and by a B-atom if $n_i = 0$. Further, let -ϵ_a, $-\epsilon_b$ and -ϵ_{ab} be the interaction energies between the $A-A, B-B$ and $A-B$ pairs. Then the total energy of a configuration having N_a A-atoms and $N - N_a$ B-atoms is given by

$$\begin{aligned} H(N_a) &= -\epsilon_a \sum_{<i,j>} n_i n_j - \epsilon_b \sum_{<i,j>} (1-n_i)(1-n_j) \\ &- \epsilon_{ab} \sum_{<i,j>} n_i(1-n_j) - \epsilon_{ab} \sum_{<i,j>} (1-n_i) n_j. \end{aligned}$$

Total number of A-atoms in the configuration is

$$N_a = \sum_i n_i.$$

Note that the hamiltonian for the mixture can, thus, be expressed in terms of the occupation number of A-atoms alone. Further, it can be easily verified that the grand partition function of the system can be expressed in terms of an effective lattice gas hamiltonian with the parameters

$$\begin{aligned} \epsilon &= \epsilon_a + \epsilon_b - 2\epsilon_{ab} \\ \mu &= \mu_a + 2z_{nn}(\epsilon_{ab} - \epsilon_b). \end{aligned}$$

Having established the equivalence of the lattice gas model (for a simple fluid or a binary mixture) with the Ising model, it is now appropriate to consider modifications of the latter.

1.2.3 n - Vector Spin Models

Earlier it was mentioned that for some examples of critical phenomena, the order parameter should have several components. Thus for planar ferromagnets, each spin variable is a two dimensional vector $\mathbf{s}_i$. The component s_i^1 (or s_i^2) varies continuously between -1 to $+1$. This lattice model, with a two-component ($n = 2$) order parameter, is generally called the $X - Y$ model. In the Heisenberg model, each spin variable is a three dimensional vector,

$$\mathbf{s}_i = (s_i^1, s_i^2, s_i^3),$$

and hence $n = 3$. More generally, one can imagine an n-vector model where

$$\mathbf{s}_i = (s_i^1, s_i^2, \cdots, s_i^n).$$

It may be noted that the dimension of the lattice on which the spins are erected can be $1,\ 2,\ 3$ or in general d. The hamiltonian is then given by

$$H(\{\mathbf{s}_i\}) = -h \sum_i s_i^1 - J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j,$$

where it is assumed that the external field is along the direction of the component s_i^1. The partition function is then to be generalized as a multiple integral

$$Z_N = \int\int d\mathbf{s}_1 \cdots d\mathbf{s}_N \exp\Big[-\frac{H}{k_B T}\Big].$$

1.2.4 Continuous Spin Models

In these models, the spin variable is regarded as an n-component vector, however, each component is allowed to take values in the range $(-\infty, \infty)$. Mathematical simplicity is the primary reason for allowing such a range for the components. However, due to the enlargement of the range of components, it is necessary to introduce a certain weight function for the components s_i^μ. Otherwise, each s_i^μ can take the value $+\infty$ and then the partition function will diverge. The standard Ising model can be regarded as a special case of this continuous spin model by properly adjusting the weight function. The hamiltonian of the continuous spin model is

$$H(\{\mathbf{s}_i\}) = -h\sum_i s_i^1 - J\sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j, -\infty < s_i^\mu < \infty.$$

Therefore, with n=1, the Ising partition function can be written as

$$Z_N = \int\int ds_1 \cdots ds_N \exp\Big[-\frac{H}{k_B T}\Big] \prod_{i=1}^{N} \{\delta(s_i + 1) + \delta(s_i - 1)\}.$$

Thus, with a weight function $W(s_i)$ defined as

$$\exp[-W(s_i)] = \delta(s_i + 1) + \delta(s_i - 1),$$

the Ising partition function can be written as

$$Z_N = \int\int ds_1 \cdots ds_N \exp\Big[-\frac{H}{k_B T} - \sum_i W(s_i)\Big].$$

This observation suggests certain simpler choices of weight functions. A model due to Kac (called the Gaussian model) uses

$$\exp[-W(\mathbf{s}_i)] = \exp(-\gamma|\mathbf{s}_i|^2), \gamma > 0.$$

Since Z_N reduces to a multiple Gaussian integral, the model can be solved exactly. However, the Gaussian model does not have a low temperature behaviour since for $T < T_c$, some critical value, the partition function diverges. A model (called the s^4 model), which is free from this difficulty, uses the weight function

$$\exp[-W(\mathbf{s}_i)] = \exp(-\gamma|\mathbf{s}_i|^2 - u|\mathbf{s}_i|^4), \gamma < 0, u > 0.$$

The quartic term in this weight function is to ensure the convergence of the integrals in the partition function. Defining an effective hamiltonian

$$H_e(\{\mathbf{s}_i\}) = H(\{\mathbf{s}_i\}) + k_B T \sum_i W(\mathbf{s}_i),$$

the partition function can be expressed as

$$Z_N = \int \int d\mathbf{s}_1 \cdots d\mathbf{s}_N \exp\Big[-\frac{H_e}{k_B T}\Big].$$

The effective hamiltonian can be rewritten in a slightly different form. Note that

$$\begin{aligned}
\sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j &= \frac{1}{2} \sum_i {\sum_j}' \mathbf{s}_i^2 + \mathbf{s}_j^2 - (\mathbf{s}_i - \mathbf{s}_j)^2, \\
&= -\frac{1}{2} \sum_i {\sum_j}' (\mathbf{s}_i - \mathbf{s}_j)^2 + z_{nn} \sum_i \mathbf{s}_i^2,
\end{aligned}$$

where $\sum_j'$ denotes summation over the nearest neighbours of i and z_{nn} is the number of nearest neighbours of any lattice point. Then H_e can be written as

$$\begin{aligned}
H_e &= a_2 \sum_i \mathbf{s}_i^2 + a_4 \sum_i \mathbf{s}_i^4 - h \sum_i s_i^1 - \frac{J}{2} \sum_i {\sum_j}' (\mathbf{s}_i - \mathbf{s}_j)^2, \\
a_2 &= \gamma k_B T - z_{nn} J \quad = a_2'(T - T_c), \quad a_2' = \gamma k_B, \\
T_c &= \frac{z_{nn} J}{\gamma k_B}, \quad a_4 = u k_B T \quad \approx a\ constant,
\end{aligned}$$

for $T \approx T_c$. The weight functions $\exp[-W(s)]$ for the various models (Figure 1.8) show that, by choosing the values of the parameters γ and u ($\gamma < 0$ and $u > 0$), it is possible to make the s^4 model resemble the Ising model. As shown in next section, the s^4 model can be derived from the Ising model in a more systematic way.

1.2.5 Kac - Hubbard - Stratonovich Transformation

Consider a general Ising hamiltonian

$$H(\{s_i\}) = -\frac{1}{2} \sum_i \sum_j J_{ij} s_i s_j.$$

Here, couplings between every pair of spins are included, however, it is assumed that $h = 0$ and $n = 1$. The partition function is

$$Z_N = \sum_{config} \exp\Big[\frac{1}{2} \sum_i \sum_j K_{ij} s_i s_j\Big],$$

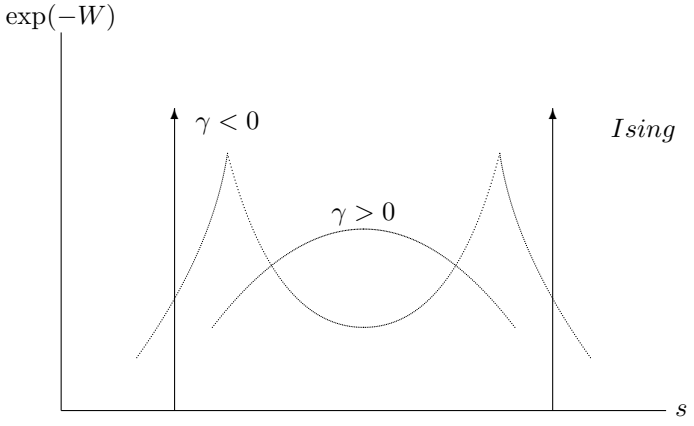

Figure 1.8: Weight Function for Various Models.

where $K_{ij} = J_{ij}/k_BT$ is non-negative and symmetric in i and j . To avoid any self interaction, it is required to put $K_{ii} = 0$. Now,

$$\frac{1}{2}\sum_i\sum_j K_{ij}s_is_j = -\frac{N}{2}p_0 + \frac{1}{2}\sum_i\sum_j P_{ij}s_is_j,$$

where $P_{ij} = p_0\delta_{ij} + K_{ij}$. By choosing p_0, the matrix $\mathbf{P}$ can be made positive definite. Hence Z_N becomes

$$Z_N = \exp\Big(-\frac{N}{2}p_0\Big)\sum_{config}\exp\Big[\frac{1}{2}(\mathbf{s},\mathbf{Ps})\Big],$$

where $\mathbf{s} = (s_1, s_2, \cdots s_N)$. If $\mathbf{Q}$ is a positive definite matrix, there is an integral representation

$$\frac{(2\pi)^{N/2}}{\sqrt{det\mathbf{Q}}} = \int_{-\infty}^{\infty}\prod_{i=1}^{N} dy_i \exp\Big[-\frac{1}{2}(\mathbf{y},\mathbf{Qy})\Big].$$

Now, introduce the vectors $\mathbf{x}$ and $\mathbf{s}$ via the transformation $\mathbf{y} = \mathbf{x} + \mathbf{Q}^{-1}\mathbf{s}$ so that

$$(\mathbf{y},\mathbf{Qy}) = (\mathbf{x},\mathbf{Qx}) + 2(\mathbf{x},\mathbf{s}) + (\mathbf{s},\mathbf{Q}^{-1}\mathbf{s}).$$

Then a more general result is

$$\frac{(2\pi)^{N/2}}{\sqrt{det\mathbf{Q}}} = \int_{-\infty}^{\infty}\prod_{i=1}^{N} dx_i \exp\Big[-\frac{1}{2}(\mathbf{x},\mathbf{Qx}) - (\mathbf{x},\mathbf{s}) - \frac{1}{2}(\mathbf{s},\mathbf{Q}^{-1}\mathbf{s})\Big].$$

Now, $\mathbf{P}$ is positive definite, so $\mathbf{P}^{-1}$ also is positive definite. Putting $\mathbf{Q} = \mathbf{P}^{-1}$ one gets

$$\frac{\sqrt{det\mathbf{P}}}{(2\pi)^{-N/2}} = \int_{-\infty}^{\infty}\prod_{i=1}^{N} dx_i \exp\Big[-\frac{1}{2}(\mathbf{x},\mathbf{P}^{-1}\mathbf{x}) - (\mathbf{x},\mathbf{s}) - \frac{1}{2}(\mathbf{s},\mathbf{Ps})\Big].$$

This equation yields

$$\exp\Big[\frac{1}{2}(\mathbf{s},\mathbf{Ps})\Big] = \frac{(2\pi)^{-N/2}}{\sqrt{detP}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \exp\Big[-\frac{1}{2}(\mathbf{x},\mathbf{P}^{-1}\mathbf{x}) - (\mathbf{x},\mathbf{s})\Big],$$

Substituting in the expression for Z_N, one finds that the configuration sum can be carried out easily. Since

$$\sum_{config} \exp[-x_i s_i] = 2\cosh(x_i),$$

the partition function becomes

$$Z_N = c \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \exp\Big[-\frac{1}{2}(\mathbf{x},\mathbf{P}^{-1}\mathbf{x}) + \sum_i \ln[2\cosh(s_i)]\Big],$$

where c is defined as

$$c = \frac{(2\pi)^{-N/2}}{\sqrt{det\mathbf{P}}} \exp(-Np_0/2).$$

Separating the diagonal part from the first term, Z_N can be expressed as

$$Z_N = c \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \exp\Big[-\frac{1}{2}\sum_i {\sum_j}' \mathbf{P}_{ij}^{-1} x_i x_j - \sum_i W(x_i)\Big],$$

where $W(x_i)$ is given by

$$W(x_i) = \frac{1}{2}\mathbf{P}_{ii}^{-1} x_i^2 - \ln[2\cosh(x_i)].$$

If $\mathbf{P}_{ij}^{-1}$ is interpreted as the coupling strength, Z_N is found to be analogous to the partition function of the continuous spin model. Expanding around $x_i = 0$ one finds

$$\begin{aligned} W(x_i) &= \gamma_i x_i^2 + u x_i^4 + O(x_i^6), \\ \gamma_i &= \frac{1}{2}(\mathbf{P}_{ii}^{-1} - 1), \quad u = \frac{5}{24}, \end{aligned}$$

which has the same form of the weight function of the continuous spin model. This mathematical equivalence between the two models allows one to conclude that the exponents obtained from them will be the same.

1.2.6 Landau - Ginzburg Model

While discussing the continuous spin model, an effective hamiltonian

$$H_e = a_2 \sum_i \mathbf{s}_i^2 + a_4 \sum_i \mathbf{s}_i^4 - h \sum_i s_i^1 - \frac{J}{2} \sum_i {\sum_j}' (\mathbf{s}_i - \mathbf{s}_j)^2,$$

was introduced. This is usually known as the (discrete) Landau-Ginzburg hamiltonian. A more physical derivation of the same is developed below. This derivation also yields a physical interpretation of the effective hamiltonian. Restricting to the case of one component spin variable, the Ising hamiltonian with general coupling constants is

$$H(\{s_i\}) = -h\sum_i s_i - \frac{1}{2}\sum_i\sum_j J_{ij}s_i s_j.$$

Now, imagine the lattice to be divided into cells of volume v. The volume is large enough so that it contains a large number (M) of lattice points, however, its linear dimension is assumed to be small compared to the correlation length. Let s'_c be the average of the spin values over the volume v, i.e.

$$s'_c = \frac{1}{M}\sum_{i\in c} s_i.$$

Thus a number s'_c can be assigned to every lattice cell. Since M is large, s'_c would vary as a continuous variable in the range $[-1, 1]$. The expression for H may be simplified by assuming that s_i in the c^{th} cell can be approximated as s'_c. Then one finds

$$H \approx -hM\sum_c s'_c - \frac{1}{2}\sum_c\sum_{i\in c}\sum_{j\in c} J_{ij}{s'_c}^2 + M\sum_c H_c^{int},$$

where H_c^{int} represents the coupling energy (per spin) of the c^{th} cell with other cells. Further, if

$$\sum_{j\in c} J_{ij} = J_0,$$

is taken to be independent of c, that is, if the lattice has translation symmetry, then H can be written as

$$H \approx -hM\sum_c s'_c - \frac{1}{2}J_0 M\sum_c {s'_c}^2 + M\sum_c H_c^{int}.$$

The values of s'_c are as such unknown quantities. If there are M^+ up spins and M^- down spins in the c^{th} cell, then

$$\begin{aligned} s'_c &= \frac{1}{M}(M^+ - M^-), \\ M &= M^+ + M^-. \end{aligned}$$

That is,

$$\begin{aligned} M^+ &= \frac{M}{2}(1 + s'_c), \\ M^- &= \frac{M}{2}(1 - s'_c). \end{aligned}$$

The entropy of the M spins is then given by $S = k_B \ln(W)$, where

$$W = \frac{M!}{M^+!\ M^-!},$$

is the total number of configurations. Now, using Sterling's approximation

$$\ln(N!) \approx N\ln(N) - N,$$

one gets

$$\begin{aligned}
\ln(W) &= \ln(M!) - \ln(M^+!) - \ln(M^-!), \\
&= M\ln(M) - M^+\ln(M^+) - M^-\ln(M^-), \\
&= -\frac{M}{2}\Big[(1+s'_c)\ln\Big(\frac{1+s'_c}{2}\Big) + (1-s'_c)\ln\Big(\frac{1-s'_c}{2}\Big)\Big],
\end{aligned}$$

where M^+ and M^- have been expressed in terms of s'_c. Taylor expansion around $s'_c = 0$ leads to

$$\ln(W) = -M\Big[-\ln(2) + \frac{1}{2}{s'_c}^{\,2} + \frac{1}{12}{s'_c}^{\,4} + \cdots\Big].$$

Therefore the free energy defined as

$$F = E - TS = H - Tk_B\sum_c \ln(W),$$

can be expressed as

$$\begin{aligned}
F &= \sum_c M[\frac{1}{2}(k_BT - J_0){s'_c}^{\,2} + \frac{1}{12}k_BT{s'_c}^{\,4} \\
&- hs'_c - k_BT\ln(2)] + M\sum_c H_c^{int}.
\end{aligned}$$

Since M spins are contained in every cell of volume v, the free energy can also be written as

$$\begin{aligned}
F &= \sum_c v\{a_0 + a_2{s'_c}^{\,2} + a_4{s'_c}^{\,4} - hs'_c\} + MH_c^{int}, \\
a_0 &= -\frac{1}{v_0}k_BT\ln(2), \\
a_2 &= \frac{1}{2v_0}(k_BT - J_0), \\
a_4 &= \frac{1}{12v_0}k_BT.
\end{aligned}$$

where v_0 is the volume associated per spin. The parameters a_0, a_2 and a_4 are functions of temperature. The term H_c^{int} representing the interaction between the cells must also be expressed in terms of s'_c. If neighbouring cells

have same average spin values, then this term should be zero. Further, the interaction between the cells should yield a positive contribution to the free energy. Therefore, a term proportional to $(s'_c - s'_{c'})^2$ may be taken as a lowest order approximation to the coupling energy between the neighbouring cells at c and c'. Thus the free energy is approximated as

$$F = \sum_c v\Big[a_0 + a_2 {s'_c}^{\,2} + a_4 {s'_c}^{\,4} - h s'_c + \frac{C}{b^2} {\sum_{c'}}' (s'_c - s'_{c'})^2\Big],$$

where C/b^2 (> 0) is a phenomenological parameter and $\sum'_{c'}$ indicates summation over the neighbours c'. The expression for F is same as that for the effective energy derived earlier. However, the above derivation uses coarse graining of the system over the linear size of the cell. Thus the effective energy of the continuous spin model is same as the free energy of the coarse grained system. If b is taken as the linear size of the cells, and since s'_c is expected to vary slowly between the cells, it is possible to go over to a continuous description where s'_c is treated as a continuous function of position. Then F becomes a functional of $s'(\mathbf{x})$,

$$F[s'] = \int_V \{a_0 + a_2 {s'}^{\,2}(\mathbf{x}) + a_4 {s'}^{\,4}(\mathbf{x}) - h s'(\mathbf{x}) + C[\nabla s'(\mathbf{x})]^2\} d\mathbf{x}.$$

This expression for F is usually known as the Landau-Ginzburg free energy functional. Even though $s'(\mathbf{x})$ is treated as a continuous function of position, it does not contain variations on a scale smaller than the cell size b. But it can describe slow variations of spin configuration over scales larger than b. Higher order terms in $\nabla s'(\mathbf{x})$ are neglected since the variation is assumed to be slow. $F[s']$ can be generalized to the case of an n-component order parameter by writing

$$\begin{aligned} {\mathbf{s}'}^{\,2} &= \mathbf{s}' \cdot \mathbf{s}' = \sum_i^n {s'_i}^{\,2}, \\ {\mathbf{s}'}^{\,4} &= \Big[{\mathbf{s}'}^{\,2}\Big]^2 \\ \Big[\nabla \mathbf{s}'\Big]^{\,2} &= \sum_\alpha^d \sum_i^n \Big[\frac{\partial s'_i}{\partial x_\alpha}\Big]^2, \end{aligned}$$

where $\alpha = 1,\ 2, \cdots\ d$ and d is the spatial dimension. In the continuum model, the subscript i denotes the spin component. However, in the lattice model, $\mathbf{s}_i$ denotes the spin variable at the i^{th} lattice site.

It is now important to see how to connect the free energy with the calculation of the partition function. Let $H'[s']$, where s' is the coarse grained order parameter, be the hamiltonian that gives the free energy of the Landau-Ginzburg model . Different configurations of s can lead to the same coarse

grained s'. Let W be the number of microscopic configurations for a specified s' in the volume v over which coarse graining has been done. Then Z, for n=1, can be written as

$$Z = \int\int \prod_c ds'_c \exp\Big[-\frac{H'[s']}{k_B T}]W.$$

Now, $k_B \ln(W) = S$ where S is the entropy of the spin variables in v and hence $W = \exp(S/k_B)$. Therefore

$$\begin{aligned} Z &= \int\int \prod_c ds'_c \exp\Big[-\frac{H'[s']}{k_B T} + \frac{TS}{k_B T}\Big], \\ &= \int\int \prod_c ds'_c \exp\Big[-\frac{F[s']}{k_B T}\Big]. \end{aligned}$$

Thus, integration over all possible variations of s', variations being over a scale greater than b, yields the total partition function.

1.3 Statistical Theory

Some general results of statistical mechanics are summarized in this section. In the Ising and n-vector models, there are a total of nL^d (n=number of spin components, d=spatial dimension) spin variables, if the linear size of the lattice is L. This is also true for the Landau-Ginzburg (L-G) model in the discrete version. According to statistical mechanics, the joint probability distribution P of these variables is

$$P = \frac{1}{Z} \exp\Big[-\frac{H[s]}{k_B T}\Big].$$

In what follows, the Boltzmann constant is taken as unity. Further, s appearing in the L-G model free energy will be the coarse grained order parameter, the symbol $'$ over s will be omitted. The free energy of the L-G model will also be called a hamiltonian. It should be noted that the hamiltonian from spin is only a part of the total hamiltonian of the system. The coupling of the spin with other modes of motion, such as lattice vibrations, and the macroscopic size of the system are the essential reasons necessitating a statistical treatment. This coupling, in fact, introduces thermal noise in the dynamics of spin variables. At thermal equilibrium, the probability distribution of the spin variables is given by the well known Boltzmann distribution. The normalization factor Z of the distribution (usually called the partition function) is given by

$$Z = \int \prod_{\mu\ \mathbf{x}} ds^\mu_{\mathbf{x}} \exp\Big[-\frac{H[s]}{T}\Big],$$

where the integral denotes the usual integral for continuous spin models and summation for discrete spin models. The notation $s^{\mu}_{\mathbf{x}}$ denotes the μ^{th} spin component at position vector $\mathbf{x}$. The free energy density F of the system is given by

$$Z = \exp\Big[-\frac{FV}{T}\Big], \quad V = L^d.$$

Thus F is a function of $T,\ h$ and other parameters (like coupling parameter J) in H. The entropy (S), magnetization (m) and specific heat (C) are given by

$$\begin{aligned} S &= -\frac{\partial F}{\partial T}, m = -\frac{\partial F}{\partial h}, \\ C &= T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}. \end{aligned}$$

The first and last are usual thermodynamic relations. The second relation can be derived easily. Writing

$$H = H_0 - h\sum_{\mathbf{x}} s^1_{\mathbf{x}},$$

since field is in direction 1, one gets

$$Z = \int \prod_{\mu}\prod_{\mathbf{x}} ds^{\mu}_{\mathbf{x}} \exp\Big[-\frac{1}{T}H_0[s] + \frac{h}{T}\sum_{\mathbf{x}} s^1_{\mathbf{x}}\Big],$$

Differentiation w.r.t h yields

$$\frac{\partial Z}{\partial h} = \frac{1}{T}\int \prod_{\mu}\prod_{\mathbf{x}} ds^{\mu}_{\mathbf{x}} \sum_{\mathbf{x}} s^1_{\mathbf{x}} \exp\Big[-\frac{1}{T}H[s]\Big].$$

Hence

$$\frac{T}{Z}\frac{\partial Z}{\partial h} = < \sum_{\mathbf{x}} s^1_{\mathbf{x}} >$$

is the total magnetization along direction 1. But then

$$\begin{aligned} F &= -\frac{T}{L^d}\ln(Z), \\ \frac{\partial F}{\partial h} &= -\frac{T}{L^d}\frac{1}{Z}\frac{\partial Z}{\partial h}. \end{aligned}$$

Hence $-\partial F/\partial h$ is same as magnetization. Now

$$\chi = \frac{\partial m}{\partial h} = -(\frac{\partial^2 F}{\partial h^2})_T,$$

where

$$\frac{\partial^2 F}{\partial h^2} = -\frac{T}{L^d}\Big[-\frac{1}{Z^2}(\frac{\partial Z}{\partial h})^2 + \frac{1}{Z}\frac{\partial^2 Z}{\partial h^2}\Big].$$

But the second derivative of Z is

$$\frac{\partial^2 Z}{\partial h^2} = \frac{1}{T^2}\int \prod_{\mu x} ds^{\mu}_{\mathbf{x}} (\sum_{\mathbf{x}} s^1_{\mathbf{x}} \sum_{\mathbf{x}'} s^1_{\mathbf{x}'}) \exp\Big[-\frac{H[s]}{T}\Big].$$

Therefore

$$\frac{T^2}{Z}\frac{\partial^2 Z}{\partial h^2} = \sum_{\mathbf{x}}\sum_{\mathbf{x}'} < s^1_{\mathbf{x}} s^1_{\mathbf{x}'} > \quad = L^d \sum_{\mathbf{x}} < s^1_{\mathbf{x}} s^1_0 > .$$

The last step follows from the translation symmetry of the system. Now, note that

$$\frac{T}{Z}\frac{\partial Z}{\partial h} = \sum_{\mathbf{x}} < s^1_{\mathbf{x}} > \quad = L^d < s^1_0 > .$$

Hence one gets

$$\Big(\frac{T}{Z}\frac{\partial Z}{\partial h}\Big)^2 = L^d L^d < s^1_0 >^2 \quad = L^d \sum_{\mathbf{x}} < s^1_{\mathbf{x}} >^2 .$$

Then χ can be expressed as

$$\chi = \frac{1}{T}\sum_{\mathbf{x}} < s^1_{\mathbf{x}} s^1_0 > - \sum_{\mathbf{x}} < s^1_{\mathbf{x}} >^2 .$$

In the continuum model, χ can be written as

$$\chi = \frac{1}{T}\int d\mathbf{x}\{< s_1(\mathbf{x}) s_1(0) > - < s_1(\mathbf{x}) >^2\} \quad = \frac{1}{T} G(0).$$

Thus χ is related to the order parameter fluctuation and its divergence is essentially due to the appearance of large spin patches neat T_c. In a similar way, the specific heat is related to fluctuations in the internal energy. The definition of F yields

$$\frac{\partial^2 F}{\partial T^2} = -\frac{2}{Z L^d}\frac{\partial Z}{\partial T} + \frac{T}{Z^2 L^d}(\frac{\partial Z}{\partial T})^2 - \frac{T}{Z L^d}\frac{\partial^2 Z}{\partial T^2}.$$

Using the derivatives of Z w.r.t T

$$\begin{aligned}
\frac{\partial Z}{\partial T} &= \frac{1}{T^2}\int \prod_{\mu x} ds^{\mu}_{\mathbf{x}} H[s] \exp\Big[-\frac{H[s]}{T}\Big], \\
\frac{\partial^2 Z}{\partial T^2} &= \int \prod_{\mu x} ds^{\mu}_{\mathbf{x}} \{-\frac{2H}{T^3} + \frac{H^2}{T^4}\} \exp\Big[-\frac{H[s]}{T}\Big],
\end{aligned}$$

one finds that

$$\begin{aligned}
\frac{1}{Z}\frac{\partial Z}{\partial T} &= \frac{1}{T^2} < H >, \\
\frac{1}{Z}\frac{\partial^2 Z}{\partial T^2} &= -\frac{2}{T^3} < H > + \frac{1}{T^4} < H^2 > .
\end{aligned}$$

Table 1.3: Exponents for 2-D Ising Model

β	$\gamma = \gamma'$	$\alpha = \alpha'$	δ	$\nu = \nu'$	η
1/8	7/4	0	15	1	1/4

Therefore C can be expressed as

$$C = -T\frac{\partial^2 F}{\partial T^2} \quad = \frac{1}{L^d T^2}(< H^2 > - < H >^2).$$

Thus specific heat is related to energy fluctuations. Near T_c, energy absorption occurs in large amounts due to complete flipping of large spin patches and this leads to large specific heat.

1.4 Summary of Exact Calculations

Having described some of the models and the method of calculations, it is appropriate to consider some results of exact calculations. However, the details of calculation are omitted.

(i) The 1-D $(d = 1)$ Ising model (as well as n-vector model) can be solved exactly. It is found that there is no spontaneous magnetization at any finite temperature. Spontaneous magnetization appears at $T = 0$ and all thermodynamic quantities diverge exponentially as $T \to 0$. For example,

$$\chi \sim x^{\gamma}, \quad x = \exp(-4J/k_B T), \quad \gamma = \frac{1}{2}.$$

When long range interaction between spins is introduced, that is,

$$J(|y - z|) \ \sim \ |y - z|^{-(2+\epsilon)}, \epsilon \ < \ 0,$$

one finds spontaneous magnetization at finite temperature, however, the exponents are different from those for short range interaction.

(ii) The 2-D $(d = 2)$ Ising model $(n = 1)$ is exactly solvable (the Onsager solution). With short range interaction, one finds spontaneous magnetization and divergence of thermodynamic quantities as T_c is approached. The critical exponents obtained are given in Table 1.3. Note that the specific heat exponent is 0 and hence the divergence is logarithmic. Another important finding is that the same exponents are obtained for all types (triangular, rectangular, hexagonal etc.) of lattices in $2 - D$.

(iii) In 3-D $(d = 3)$, the Ising model $(n = 1)$ has not been solved exactly. However, very accurate estimates of exponents are available from the high

Table 1.4: Exponents for 3-D Ising Model

β	γ	α	δ	ν
0.328 ± 0.008	1.239 ± 0.002	0.105 ± 0.01	5.0 ± 0.05	0.632 ± 0.002

temperature series expansion method. They are given in Table 1.4. The universal aspects of critical exponents in $3 - D$ are also well established.

(iv) The Gaussian model is exactly solvable. Since it does not have a low temperature behaviour, the exponent β is unspecified. Others are $\gamma = 1$, $\alpha = (4 - d)/2$ for $d \leq 4$ and $\alpha = 0$ for $d > 4, \delta = 3, \nu = 1/2$ and $\eta = 0$. These results show that critical exponents depend strongly on the spatial dimension.

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Chapter 2

Landau's Theory and Gaussian Fluctuations

In this chapter, the partition function and the exponents, which characterize the divergence of thermodynamic quantities, are calculated using a linearised version of the Landau-Ginzburg (L-G) energy functional. The partition function in the L-G model can be written as a functional integral

$$Z = \int \cdots \int \prod_i Ds_i(\mathbf{x}) \exp\Big[- \frac{H[s]}{T} \Big].$$

Recall that $s_i(\mathbf{x})$ is the i^{th} component of the order parameter field. The functional integral is a notation which indicates that all the possible variations of the order parameter field have to be accounted in calculating the partition function. In the continuum model, the hamiltonian $H[s]$ was obtained as

$$H[s] = \int_V \{a_0 + a_2 \mathbf{s}^2(\mathbf{x}) + a_4 \mathbf{s}^4(\mathbf{x}) - h s_1(\mathbf{x}) + c[\nabla \mathbf{s}(\mathbf{x})]^2\} d\mathbf{x},$$

where

$$a_0 = -k_B T \ln(2), \quad a_2 = a_2'(T - T_c), \quad a_4 = u k_B T,$$

and c is a phenomenological parameter characterizing the spatial variation of $\mathbf{s}$. Since H/T ($k_B = 1$) appears in the Boltzmann factor, it is appropriate to write

$$\frac{H[s]}{T} = \int_V \Big[a_0^* + a_2^* \mathbf{s}^2(\mathbf{x}) + a_4^* \mathbf{s}^4(\mathbf{x}) - h^* s_1(\mathbf{x}) + c^* (\nabla \mathbf{s}(\mathbf{x}))^2 \Big] d\mathbf{x},$$

where $a_j^* = a_j/T$, $h^* = h/T$, and $c^* = c/T$. Then, note that

$$a_2^* = \frac{a_2'}{T_c}(T - T_c) = a_2'^{\,*}(T - T_c),$$

and all the other parameters in $H[s]/T$ can be approximated as constants near T_c. This approximation does not affect the values of exponents since they are defined in the limit T approaching T_c. Hereafter, the symbol $*$ on the parameters in $H[s]/T$ will be omitted.

The spatial variation of $s_i(\mathbf{x})$ does not contain length scales below a cut-off value b, which characterizes the coarse graining length. Thus a Fourier expansion of $s_i(\mathbf{x})$ should be written as

$$s_i(\mathbf{x}) = \frac{1}{L^{d/2}} \sum_{k \le \Lambda} \exp(\imath \mathbf{k} \cdot \mathbf{x}) s_{i\mathbf{k}}, \quad 1 \le i \le n,$$

where $\Lambda = 2\pi/b$ is the cut-off wave number. The Fourier component $s_{i\mathbf{k}}$ is given by

$$s_{i\mathbf{k}} = \frac{1}{L^{d/2}} \int \exp(-\imath \mathbf{k} \cdot \mathbf{x}) s_i(\mathbf{x}) d\mathbf{x}, \quad \le i \le n.$$

The orthogonality of the Fourier modes yields

$$\begin{aligned} \frac{H[s]}{T} &= a_0 L^d + \sum_i \sum_{k \le \Lambda} (a_2 + ck^2) s_{i\mathbf{k}} s_{i\ -\mathbf{k}} - \frac{h}{L^{d/2}} s_{i\ 0} \\ &+ \frac{a_4}{L^d} \sum_{ij} \sum_{k,k',k'' \le \Lambda} s_{i\mathbf{k}} s_{i\mathbf{k}'} s_{j\mathbf{k}''} s_{j\ -\mathbf{k}-\mathbf{k}'-\mathbf{k}''}. \end{aligned}$$

Then, the probability distribution of the Fourier amplitudes is

$$P(\{s_{i\mathbf{k}}\}) = \frac{1}{Z} \exp\Big[- \frac{H[s]}{T} \Big].$$

2.1 Landau's Theory

In Landau's theory, the spin distribution is obtained by minimizing $H[s]$ which amounts to maximizing $P[s]$. Thus one deals with the most probable spin distribution in the system. It can be easily seen that the most probable distribution must be spatially uniform. To show this, let $\mathbf{s}(\mathbf{x})$ be written as

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}'' + \mathbf{s}'(\mathbf{x}),$$

where $\mathbf{s}'$ is spatially constant. Such a separation, with the additional condition that $\int \mathbf{s}' d\mathbf{x} = 0$, is always possible. With this substitution, the energy density becomes

$$\begin{aligned} H_d[s] &= a_0 + a_2 \mathbf{s}^2 + a_4 \mathbf{s}^4 - h s_1 + c(\nabla \mathbf{s})^2 \\ &= [a_0 + a_2 \mathbf{s}''^{\,2} + a_4 \mathbf{s}'^{\,4} - h s_1''] + [2a_2 \mathbf{s}'' \cdot \mathbf{s}' + 4a_4 \mathbf{s}''^{\,2} (\mathbf{s}'' \cdot \mathbf{s}') - h s_1''] \end{aligned}$$

$$+ [a_2 \mathbf{s}'^{\,2} + 2a_4 \mathbf{s}''^{\,2} \mathbf{s}'^{\,2} + 4a_4 (\mathbf{s}'' \cdot \mathbf{s}')^2 + 4a_4 \mathbf{s}'^{\,2} (\mathbf{s}'' \cdot \mathbf{s}') + a_4 \mathbf{s}'^{\,4} + c(\nabla \mathbf{s}')^2].$$

The volume integral of the second square bracket is zero. The $3^{rd}, 4^{th}$ and 5^{th} terms in the last square bracket can be combined together to yield

$$\frac{H[s]}{T} = \frac{H[s'']}{T} + \int \Big[(a_2 + 2a_4 \mathbf{s}''^{\,2})\mathbf{s}'^{\,2} + a_4(\mathbf{s}'^{\,2} + 2\mathbf{s}'' \cdot \mathbf{s}')^2 + c(\nabla \mathbf{s}')^2\Big] d\mathbf{x}.$$

The integral term, which vanishes when $\mathbf{s}' = 0$, definitely gives a positive contribution if $w = (a_2 + 2a_4 \mathbf{s}''^{\,2}) > 0$. Now, $w \geq 0$ if $a_2 \geq 0$, however, as shown below, $\mathbf{s}''$ can be chosen such that $w \geq 0$ even when $a_2 < 0$. Therefore, the most probable distribution should be spatially constant. Now, minimizing $H[s'']$ one gets

$$2s_i''(a_2 + 2a_4 \mathbf{s}''^{\,2}) = h\delta_{i1}, \quad 1 \leq i \leq n.$$

When $h = 0$, the solutions are

$$\mathbf{s}'' = 0, \quad |\mathbf{s}''| = \sqrt{\frac{-a_2}{2a_4}}.$$

From the nature of $H[s'']$ (Figure 2.1), one notes that the solution $\mathbf{s}'' = 0$ corresponds to the case $a_2 \geq 0$ and the other solution is for $a_2 < 0$. One also finds that for $a_2 < 0$, only $|\mathbf{s}''|$ is determined, there by showing that there are infinite number of solutions. When $h \neq 0$, the form of $H[s'']$ shows that

$$s_i'' = 0, \quad 2 \leq i \leq n,$$

and the component along h satisfies the equation

$$2s_1''[a_2 + 2a_4 s_1''^{\,2}] = h. \tag{2.1}$$

Now, $a_2 = a_2'(T - T_c)$ and the other parameters are constants near T_c. Thus for $h = 0$, the solutions are

$$\begin{aligned} \mathbf{s}'' &= 0, \quad T > T_c \\ |\mathbf{s}''| &= \sqrt{\frac{a_2'}{2a_4}}(T_c - T)^{1/2}, \quad T < T_c. \end{aligned}$$

The temperature dependence of $|\mathbf{s}''|$ below T_c shows that the order parameter exponent is $\beta = 1/2$. Keeping up to quadratic terms, these solutions also show that

$$\begin{aligned} \frac{H[s'']}{T} &= L^d a_0, \; T > T_c, \\ \frac{H[s'']}{T} &= L^d[a_0 - \frac{a_2'^{\,2}}{2a_4}(T - T_c)^2], \; T < T_c. \end{aligned}$$

Thus, when $h = 0$, the free energy density $H[s'']/L^d$ is

$$\begin{aligned} F &= a_0 T, \quad T > T_c, \\ &= T[a_0 - \frac{a_2'^{\,2}}{2a_4}(T - T_c)^2], \quad T < T_c. \end{aligned}$$

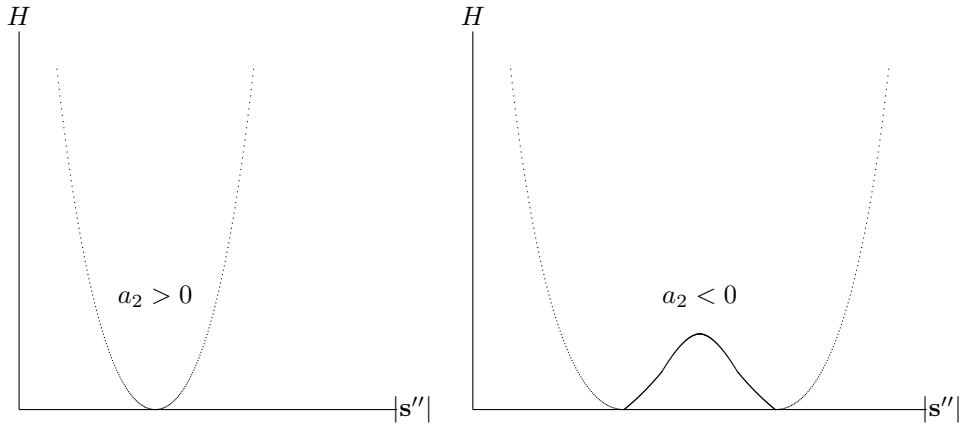

Figure 2.1: Hamiltonian Vs $|\mathbf{s}''|$.

Therefore, there is a discontinuity in specific heat across T_c and it is given by

$$\triangle C = -T\frac{\partial^2 \triangle F}{\partial T^2} \quad = T_c^2 \frac{{a_2'}^2}{a_4} \; at \; T = T_c.$$

A discontinuity in specific heat shows that the specific heat exponent $\alpha = 0$. At $T = T_c, a_2 = 0$. Then Eq.(2.1) shows that

$$s_1''^{\,3} = \frac{h}{4a_4}.$$

Therefore, the critical isotherm exponent $\delta = 3$. To obtain s_1'' with a small non-zero h, rewrite Eq.(2.1) for $a_2 > 0$ as

$$s_1'' = \frac{h}{a_2 + 2a_4 s_1''^{\,2}} \approx \frac{h}{2a_2} = \frac{h}{2a_2'(T - T_c)}, \; T > T_c.$$

To get an approximation when $a_2 < 0$, rewrite Eq.(2.1) as

$$[-m_0^2 + s_1''^{\,2}] = (-m_0 + s_1'')(m_0 + s_1'') = \frac{h}{4a_4 s_1''},$$

where $m_0^2 = |a_2|/2a_4$. That is

$$s_1'' = m_0 + \frac{h}{4a_4 s_1''(m_0 + s_1'')}.$$

Now, substituting $s_1'' \approx m_0$ on the r.h.s, one gets

$$s_1'' \approx m_0 + \frac{h}{8a_4 m_0^2} = \sqrt{\frac{a_2'}{2a_4}}(T_c - T)^2 + \frac{h}{4a_2'(T_c - T)} \; , \; T < T_c.$$

Thus, when $T > T_c$, $s_1'' \to 0$ as $h \to 0$, while it approaches a finite value for $T < T_c$. Now, the susceptibility can be calculated as

$$\chi = (\frac{\partial s_1''}{\partial h})_T = \begin{cases} 1/[2a_2' T_c(T - T_c)], & T > T_c \\ 1/[4a_2' T_c(T_c - T)], & T < T_c. \end{cases}$$

Note that the parameter h in s_1'' actually represents h/T_c. Thus, the susceptibility exponents are $\gamma = \gamma' = 1$. The exponents given by Landau's theory are same as those obtained in the mean field solution of Ising model, which is discussed in many standard text books (for eg. K. Huang, Statistical Mechanics). Since the spatial variation of the order parameter is not accounted in these approaches, they can not provide any information on the exponents ν and η.

2.2 Gaussian Approximation

The spatial variation of the spin field can be incorporated in the calculation of the partition function in an approximate way. Exact calculations can not be done due to the occurrence of quartic terms in the hamiltonian. Therefore, these terms are approximated by assuming that the spatial variation of the spin filed is a perturbation. That is, $\mathbf{s}(\mathbf{x})$ is written as $\mathbf{s}(\mathbf{x}) = \mathbf{s}'' + \mathbf{s}'(\mathbf{x})$, where $\mathbf{s}''$ is the most probable value of $\mathbf{s}$ given by Landau's theory, and $\mathbf{s}'(\mathbf{x})$ is a 'small' correction for the spatial dependence. Then the L-G hamiltonian becomes

$$\frac{H[s]}{T} = \frac{H[s'']}{T} + \int \Big[(a_2 + 2a_4 \mathbf{s}''^{\,2})\mathbf{s}'^{\,2} + a_4(\mathbf{s}'^{\,2} + 2\mathbf{s}'' \cdot \mathbf{s}')^2 + c(\nabla \mathbf{s}')^2\Big] d\mathbf{x}.$$

Now, terms beyond quadratic terms in $\mathbf{s}'$ are omitted to obtain

$$\frac{H[s]}{T} \approx \frac{H[s'']}{T} + \int \Big[(a_2 + 2a_4 \mathbf{s}''^{\,2})\mathbf{s}'^{\,2} + 4a_4(\mathbf{s}'' \cdot \mathbf{s}')^2 + c(\nabla \mathbf{s}')^2\Big] d\mathbf{x}.$$

Let h be in the direction $i = 1$ so that $\mathbf{s}''$ is also along $i = 1$. Then the above expression can be written as

$$\begin{aligned} \frac{H[s]}{T} &= \frac{H[s_1'']}{T} + \int \Big[\{a_2 + 6a_4 s_1''^{\,2}\}s_1'^{\,2} \\ &+ c(\nabla s_1')^2 + \{a_2 + 2a_4 s_1''^{\,2}\}\mathbf{s}_+'^{\,2} + c(\nabla \mathbf{s}_+')^2\Big] d\mathbf{x}. \end{aligned}$$

where $\mathbf{s}_+'$ is along a direction perpendicular to h, i.e. it has components along $i = 2, 3, \cdots n$. In terms of Fourier components $\{s_{i\mathbf{k}}\}$, one gets

$$\begin{aligned} \frac{H[s]}{T} &= \frac{H[s_1'']}{T} + \sum_{k \neq 0} \Big[a_2 + 6a_4 s_1''^{\,2} + ck^2\Big] |s_{1k}|^2 \\ &+ \sum_{i=2}^{n} \sum_{k \neq 0} \Big[a_2 + 2a_4 s_1''^{\,2} + ck^2\Big] |s_{i\mathbf{k}}|^2. \end{aligned} \tag{2.2}$$

Thus, a quadratic approximation to $H[s]/T$, which includes the space dependent part, has been obtained.

2.2.1 Above T_c

Consider the case $T > T_c$ and $h = 0$. Even though $h = 0$, the susceptibility can be computed from fluctuations in $s_i(\mathbf{x})$. If $T > T_c$ and $h = 0$, the most probable value $\mathbf{s}''$ is 0. Therefore the quadratic approximation reduces to

$$\frac{H[s]}{T} = a_0 L^d + \sum_i \sum_{k\neq 0} (a_2 + ck^2)|s_{i\mathbf{k}}|^2, \tag{2.3}$$

where the contribution from $H[s_1'']/T$ is shown explicitly. Since this expression contains only quadratic terms in $s_{i\mathbf{k}}$, their probability distribution is Gaussian. Now $< s_{i\mathbf{k}} >$, $< s_{i\mathbf{k}}^2 >$ and the free energy density F are to be calculated. As noted earlier, the probability distribution of $s_{i\mathbf{k}}$ is

$$P(\{s_{i\mathbf{k}}\}) = \frac{1}{Z} \exp\Big[- L^d a_0 - \sum_{i\ k\neq 0} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 \Big].$$

Normalization of P yields

$$Z \exp(L^d a_0) = \int \prod_{ik\leq\Lambda} ds_{i\mathbf{k}} \exp\Big[- \sum_{i\ k\neq 0} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 \Big].$$

Note that $s_{i\mathbf{k}}$ and $s_{i\ -\mathbf{k}}$ are complex conjugates since $s_i(\mathbf{x})$ is a real function. Therefore, the last expression is rewritten as

$$Z \exp(L^d a_0) = \int (\prod_{i\ k'\leq\Lambda} ds_{i\mathbf{k}'} ds_{i\ -\mathbf{k}'}) \exp\Big[- 2 \sum_{i\ k'\neq 0} (a_2 + ck'^{\,2})|s_{i\mathbf{k}'}|^2 \Big].$$

As shown in Figure 2.2 (for a two dimensional case), $\mathbf{k}'$ is a vector in the shaded region. Now,

$$ds_{i\mathbf{k}'} ds_{i\ -\mathbf{k}'} = 2 ds_{i\mathbf{k}'}^R ds_{i\mathbf{k}'}^I,$$

where the superscripts R and I denote the real and imaginary parts respectively. Therefore the expression for Z becomes

$$Z \exp(L^d a_0) = \int \prod_{i\ k'\leq\Lambda} 2 ds_{i\mathbf{k}'}^R ds_{i\mathbf{k}'}^I \exp\Big[- 2 \sum_{i\ k'\neq 0} (a_2 + ck'^{\,2})(s_{i\mathbf{k}'}^{R\,2} + s_{i\mathbf{k}'}^{I\,2}) \Big].$$

Now, each of the integrals on the r.h.s can be calculated to obtain

$$Z \exp(L^d a_0) = \prod_{i\ k'\leq\Lambda} \frac{\pi}{(a_2 + ck'^{\,2})} = \Big[\prod_{i\ k\leq\Lambda} \frac{\pi}{(a_2 + ck^2)} \Big]^{1/2}.$$

Free energy density F defined as

$$Z = \exp\Big[- \frac{F L^d}{T} \Big],$$

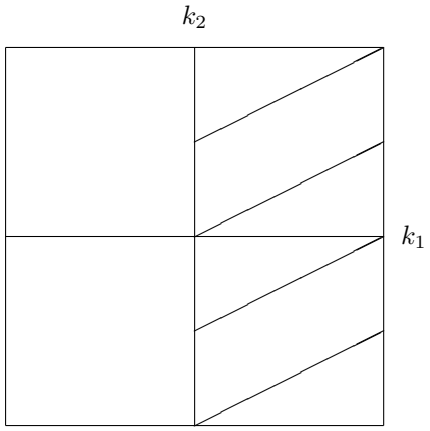

Figure 2.2: $k_1 - k_2$ Plane.

can now be readily obtained and the result is

$$\begin{aligned} F &= a_0 T - \frac{T}{2L^d} \sum_i \sum_{k \leq \Lambda} \ln \Big[\frac{\pi}{a_2 + ck^2} \Big] \\ &= a_0 T - \frac{T}{2L^d} n \sum_{k \leq \Lambda} \ln \Big[\frac{\pi}{a_2 + ck^2} \Big]. \end{aligned} \tag{2.4}$$

where n denotes the number of components of the order parameter. Exactly similar calculations can be done to obtain the averages. For instance, the average of $s_{i\mathbf{k}}$, which is defined as

$$< s_{i\mathbf{k}} >= \int \prod_i \prod_{k \leq \Lambda} ds_{i\mathbf{k}} \; s_{i\mathbf{k}} P(\{s_{i\mathbf{k}}\}),$$

is zero. More generally one finds that $< s_{i\mathbf{k}} s_{j\mathbf{k}} >= 0$ for $i \neq j$. The Fourier transform of correlation function is given by

$$G(k) =< |s_{i\mathbf{k}}|^2 > \quad = \int \prod_i \prod_{k \leq \Lambda} ds_{i\mathbf{k}} \; |s_{i\mathbf{k}}|^2 P(\{s_{i\mathbf{k}}\}) = \frac{1}{2(a_2 + ck^2)}.$$

Note that,

$$\mathbf{m} =< \mathbf{s} >=< \mathbf{s}'' + \mathbf{s}' >=< \mathbf{s}'' >,$$

which is same as the most probable value. Hence the exponents β and δ, which relate m to T and h, are the same as those ($\beta = 1/2, \delta = 3$) given by Landau's theory. Now, the susceptibility χ can be obtained from $G(k)$ as

$$\chi = \frac{G(0)}{T_c} = \frac{1}{2T_c a_2} = \frac{1}{2T_c a_2'} (T - T_c)^{-1},$$

and hence the exponent $\gamma = 1$. In other words, addition of Gaussian fluctuations does not affect these exponents. However, one finds that

$$G(k) \sim k^{-2} \; at \; T = T_c,$$

since $a_2(T_c) = 0$. This result shows that the exponent $\eta = 0$. The characteristic length ξ (coefficient of k) in the correlation function is given by

$$\xi = \sqrt{\frac{c}{a_2}} = \sqrt{\frac{c}{a_2'}}(T - T_c)^{-1/2}.$$

Therefore the correlation length exponent $\nu = 1/2$. The spatial correlation function $G(r)$ can be obtained by taking the inverse transform of $G(k)$,

$$\begin{aligned} G(r) &= \frac{1}{(2\pi)^d}\int \exp(\imath \mathbf{k}\cdot\mathbf{r})G(k)d\mathbf{k} \\ &= \frac{1}{(2\pi)^d}\frac{\xi^2}{2c}\int \exp(\imath \mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{k}}{1+\xi^2k^2} \\ &= \frac{1}{(2\pi)^d}\frac{\xi^{2-d}}{2c}\int \exp(\imath \mathbf{k}'\cdot\mathbf{r}/\xi)\frac{d\mathbf{k}'}{1+k'^{\,2}}. \end{aligned}$$

In the last step, the substitution $\mathbf{k}' = \xi\mathbf{k}$ has been made. For T close to T_c, $\xi \to \infty$ and the $\mathbf{k}'$ integration can be extended over the full k-space to obtain

$$G(r) \approx \frac{1}{r}\exp\Big[-\frac{r}{\xi}\Big] \; for \; d = 3.$$

Thus the spatial correlation function falls of exponentially and ξ is the characteristic length.

Finally, to obtain the specific heat and the exponent α, consider the expression for F in Eq.(2.4). There, a_0 is a smooth function of T and hence the divergence in specific heat can arise only from the logarithmic term when $k \approx 0$. First of all, consider the limit of L tending to infinity, i.e. the thermodynamic limit. Now, each component $k_i = 2\pi n_i/L$, so the number of wave vectors in $d\mathbf{k}$ is $(L/2\pi)^d d\mathbf{k}$. In other words, the density of modes, each mode being denoted by a wave vector $\mathbf{k}$, is $(L/2\pi)^d$. Therefore, in the limit of large L, the sum over k can be replaced as

$$\sum_{k\le\Lambda} \to (\frac{L}{2\pi})^d \int d\mathbf{k}.$$

Thus Eq.(2.4) becomes

$$F = a_0 T - \frac{T}{2}\frac{n}{(2\pi)^d}\int_0^\Lambda \ln\Big[\frac{\pi}{a_2 + ck^2}d\mathbf{k}\Big].$$

Now, the specific heat $C = -T\partial^2 F/\partial T^2$ is given by

$$C = \frac{n}{2} a_2'^{\,2} T^2 \frac{1}{(2\pi)^d} \int_0^\Lambda \frac{d\mathbf{k}}{(a_2 + ck^2)^2} - \frac{n a_2' T}{(2\pi)^d} \int_0^\Lambda \frac{d\mathbf{k}}{a_2 + ck^2}.$$

As T approaches T_c, a_2 tends to zero. Then the first integral on the r.h.s diverges at lower limit for $d \leq 4$. The second integral diverges for $d \leq 2$. For $d > 4$, both integrals are finite and hence there is no divergence in C as T_c is approached. In other words, for $d > 4, C$ has only a jump discontinuity as given by Landau's theory and the exponent $\alpha = 0$. Making a change of variable $\mathbf{k} = \mathbf{k}'/\xi$, where

$$\xi^{-2} = \frac{a_2}{c} = \frac{a_2'}{c}(T - T_c),$$

C can be expressed as

$$C = \frac{n}{2}\Big[\frac{a_2' T}{c}\Big]^2 \xi^{4-d} \kappa_d I_1(\xi\Lambda) - \frac{n a_2' T}{c} \xi^{2-d} \kappa_d I_2(\xi\Lambda).$$

Here κ_d is $(2\pi)^{-d}$ times the angular part of the d-dimensional integral and I_1 and I_2 are given by

$$\begin{aligned} I_1(\xi\Lambda) &= \int_0^{\xi\Lambda} \frac{k'^{\,d-1} dk'}{(1 + k'^{\,2})^2} \\ I_2(\xi\Lambda) &= \int_0^{\xi\Lambda} \frac{k'^{\,d-1} dk'}{1 + k'^{\,2}}. \end{aligned}$$

Now, for $2 < d < 4$, as $\xi \to \infty, I_1(\infty)$ remains finite. That is,

$$I_1(\infty) = \int_0^\infty \frac{x^{d-1} dx}{(1 + x^2)^2} = \frac{\pi - \pi d/2}{\sin(\pi d/2)}.$$

I_2 can be written as

$$\begin{aligned} I_2(\xi\Lambda) &= \int_0^{\xi\Lambda} x^{d-3}[1 - \frac{1}{1 + x^2}]dx \\ &= \frac{(\xi\Lambda)^{d-2}}{d-2} - \int_0^{\xi\Lambda} \frac{1}{1 + x^2} \frac{dx^{d-2}}{d-2} \\ &= \frac{(\xi\Lambda)^{d-2}}{d-2} - \frac{1}{d-2}\Big[\frac{(\xi\Lambda)^{d-2}}{1 + \xi^2\Lambda^2} + 2I_1(\xi\Lambda)\Big]. \end{aligned}$$

Therefore one finds that

$$\xi^{2-d} I_2(\xi\Lambda) \approx \frac{\Lambda^{d-2}}{d-2},$$

for large ξ. The temperature dependence of C near T_c is

$$\begin{aligned} C &= C_0 \xi^{4-d} + C_1 \\ &= C_0 (T - T_c)^{(2-d)/2} + C_1, \end{aligned}$$

where C_0 and C_1 are some constants and $2 < d < 4$. Thus α is found to be

$$\alpha = \frac{4-d}{2} \; for \; 2 < d < 4.$$

For $d = 4$ one gets

$$\begin{aligned} I_1(\xi\Lambda) &= \int_0^{\xi\Lambda} \frac{x^3 dx}{(1+x^2)^2} = -\frac{1}{2} \int_0^{\xi\Lambda} y d(1+y)^{-1} \\ &= \frac{1}{2}\Big[\ln(\xi\Lambda)^2 - 1 + O(\xi\Lambda)^{-2} \Big], \\ I_2(\xi\Lambda) &= \int_0^{\xi\Lambda} \frac{x^3 dx}{1+x^2} = \frac{1}{2} \int_0^{\xi\Lambda} \frac{y dy}{1+y} \\ &= \frac{1}{2}\Big[\xi^2\Lambda^2 - \ln(1+\xi^2\Lambda^2) \Big]. \end{aligned}$$

Therefore, it is easily found that

$$C = C_0 \ln(T - T_c) + C_1,$$

for $d = 4$ and so $\alpha = 0$. Thus, these calculations show that the value of α is not modified by the addition of Gaussian fluctuations for $d \geq 4$. However, for $d < 4$, the specific heat is found to diverge as T_c is approached. Thus it may be said that Landau's theory is consistent, i.e. the omission of spatial fluctuations is appropriate only for $d \geq 4$. Some of the other important points to be noted are the following. (i) The exponents obtained are independent of the details $(a_2', c, etc.)$ of the hamiltonian as in Landau's theory, and thus they show universality. (ii) Singular behavior of thermodynamic quantities arises from a hamiltonian, with regular parameters, which was obtained by coarse graining over small length scales. (iii) The spatial dimension enters into the exponent α, however, all the exponents are independent of the order parameter dimension n.

2.2.2 Below T_c

Now consider the low temperature case when $\mathbf{s}'' \neq 0$ even when $h = 0$. Assuming h to be along direction 1, the expression for s_1'' in Eq.(2.1) implies

that

$$2a_4 s_1''^{\,2} = \frac{h}{2s_1''} - a_2.$$

Further, one also finds

$$a_2 + 6a_4 s_1''^{\,2} = \frac{3h}{2s_1''} - 2a_2 = \frac{3h}{2s_1''} + 2a_2'(T_c - T).$$

Therefore, Eq.(2.2) for the linearised hamiltonian becomes

$$\begin{aligned} \frac{H[s]}{T} &= \frac{H[s_1'']}{T} + \sum_{k\leq\Lambda} \Big[\frac{3h}{2s_1''} + 2a_2'(T_c - T) + ck^2\Big]|s_{1\ \mathbf{k}}|^2 \\ &+ \sum_{i=2}^{n}\sum_{k\leq\Lambda} \Big[\frac{h}{2s_1''} + ck^2\Big]|s_{i\mathbf{k}}|^2. \end{aligned}$$

This is exactly of the form as in Eq.(2.3) for the case $T > T_c$. Hence the free energy density is

$$\begin{aligned} F &= \frac{H[s_1'']}{L^d} - \frac{T}{2L^d}\sum_{k\leq\Lambda} \ln\Big[\frac{\pi}{3h/(2s_1'') + 2a_2'(T_c - T) + ck^2}\Big] \\ &- \frac{(n-1)T}{2}\sum_{k\leq\Lambda} \ln\Big[\frac{\pi}{h/(2s_1'') + ck^2}\Big]. \end{aligned} \tag{2.5}$$

The Fourier transform of the correlation functions are given by

$$\begin{aligned} G_1(k) &= <|s_{1\mathbf{k}}|^2> \ = \frac{1}{2}\frac{1}{3h/(2s_1'') + 2a_2'(T_c - T) + ck^2}, \\ G_+(k) &= <|s_{i\mathbf{k}}|^2> \ = \frac{1}{2}\frac{1}{h/(2s_1'') + ck^2}, \quad 2 \leq i \leq n. \end{aligned}$$

Thus the longitudinal ($i = 1$) and the transverse ($2 \leq i \leq n$) parts of the correlation function are found to have different behavior. From the expression for $G_1(k)$ with $h = 0$, the characteristic length is found to be

$$\xi' = (\frac{c}{2a_2'})^{1/2}(T_c - T)^{-1/2},$$

and so the exponent $\nu' = 1/2$. Putting $T = T_c$ yields $G_1(k) = k^2$ and hence $\eta' = 0$. Further, χ is found to be

$$\chi = \frac{G_1(0)}{T_c} \ = \frac{1}{4T_c a_2'(T_c - T)},$$

which shows that the exponent $\gamma' = 1$. As in the case of $T > T_c$, the spatial correlation function can be shown to be

$$G(r) \approx \frac{1}{r}\exp\Big[-r/\xi'\Big] \ for\ d = 3.$$

The transverse part of susceptibility is

$$\chi_+ = \frac{G_+(0)}{T_c} \quad = \frac{s_1''}{hT_c},$$

which diverges as $h \to 0$. This unphysical divergence is a drawback of the Gaussian approximation and this aspect is further discussed below.

For calculating the specific heat exponent, assume that $h = 0$. Then taking the thermodynamic limit, Eq.(2.5) for the free energy density becomes

$$\begin{aligned} F &= F_L - \frac{T}{(2\pi)^d} \int_0^\Lambda \ln\Big[\frac{\pi}{2a_2'(T_c - T) + ck^2}\Big] d\mathbf{k} \\ &- \frac{T(n-1)}{2} \frac{1}{(2\pi)^d} \int_0^\Lambda \ln\Big[\frac{\pi}{ck^2}\Big] d\mathbf{k}. \end{aligned}$$

The first term F_L gives only a jump in the specific heat C. The last term does not contribute to C since it is linear in T. The second term is similar to that obtained for $T > T_c$, the differences are that the term n is absent and a_2' is replaced by $2a_2'$. Therefore the specific heat C is

$$C = \frac{1}{2} \frac{(2a_2'T)^2}{(2\pi)^d} \int_0^\Lambda \frac{d\mathbf{k}}{2a_2'(T_c - T) + ck^2} - \frac{2a_2'T}{(2\pi)^d} \int_0^\Lambda \frac{d\mathbf{k}}{2a_2'(T_c - T) + ck^2}.$$

Then, the integrals can be analyzed exactly as before and the result is

$$\begin{aligned} C &= C_0' \xi'^{\,4-d} + C_1' \\ &= C_0' (T_c - T)^{-(4-d)/2} + C_1', \end{aligned}$$

where ξ' is the correlation length for $T < T_c$ and $2 < d < 4$. Thus one gets

$$\alpha' = \begin{cases} (4-d)/2 & for \quad 2 < d < 4 \\ 0 & for \quad d \geq 4, \end{cases}$$

Once again it is found that the exponents of Landau's theory are not modified by the addition of Gaussian fluctuations for $d \geq 4$.

2.3 Fluctuations and Dimension

Earlier, it was observed that the Gaussian approximation is a consistent approach for $d \geq 4$. Thus, fluctuations (or at least their contributions to specific heat) are negligible for $d \geq 4$ and the Gaussian approximation seems meaningful. As d becomes less than 4, fluctuations become more and more important. In fact it is known that for $d = 1$ and $n = 1$, fluctuations are so

strong that $< s >= 0$ for any finite T. Similar results hold for $d = 2$ and $n \geq 2$ also. To give a qualitative argument for this fact, consider the transverse part of the correlation function for $T < T_c$,

$$\begin{aligned} G_+(r) &= (2\pi)^{-d} \int \exp(\imath \mathbf{k} \cdot \mathbf{r}) G_+(k) d\mathbf{k} \\ &= (2\pi)^{-d} \int_0^{\Lambda} \exp(\imath \mathbf{k} \cdot \mathbf{r}) \frac{k^{d-1} dk}{h/ < \mathbf{s} > +2ck^2}. \end{aligned}$$

The l.h.s is a finite quantity since it is the average of spin components which are bounded variables. For $d \leq 2$, the lower limit contribution for the r.h.s diverges as $h \to 0$. This unphysical result is a consequence of the assumptions of Landau's theory with Gaussian fluctuations. Divergence is absent if $< \mathbf{s} >$ vanishes as $h \to 0$. Note that for $T < T_c$ and $h = 0$, there are an infinity of configurations having approximately the same energy when $n \geq 2$, i.e. when G_+ comes into picture. The most probable spin profile is spatially uniform when $h = 0$. However, $\mathbf{s}(\mathbf{x})$ can change the direction very slowly and such a configuration will have only slightly higher total energy due to the $(\nabla \mathbf{s})^2$ term. In Landau's approach of considering only the most probable configuration, configurations of almost same probability can not be accounted for. But, it can be argued that the infinite number of configurations (of approximately the same energy and hence same probability of occurrence), arising from all possible directions of the spin variable, make $< \mathbf{s} >= 0$. These facts can not be incorporated in Landau's approach and it should be concluded that it fails completely for $d \leq 2$. Fluctuations become very predominant for $d \leq 2$ and they can not be treated within the framework of Landau's theory.

2.4 Adequacy of Gaussian Approximation

The results of Landau's theory are based on the assumption that the partition function can be calculated by taking the dominant contribution of the functional integral defining it. Spatial fluctuations in the order parameter are, then, accounted by linearising the hamiltonian around the dominant contribution. This approach leads to a Gaussian probability distribution for the Fourier amplitudes of the order parameter. It was argued that this procedure is consistent for spatial dimension $d \geq 4$. To establish this point further, consider the partition function

$$Z = \exp(-a_0 V) \int Ds \exp\Big[- \int_V \{a_2 s^2(\mathbf{x}) + a_4 s^4(\mathbf{x}) + c[\nabla s(\mathbf{x})]^2\} d\mathbf{x} \Big].$$

For simplicity, it is assumed that $T > T_c,\ h = 0$ and $n = 1$. The argument in the exponential can be rewritten with the transformations

$$\begin{aligned} \mathbf{x} &\rightarrow \sqrt{\frac{c}{a_2}}\mathbf{x}', \quad \nabla_{\mathbf{x}} \rightarrow \sqrt{\frac{a_2}{c}}\nabla_{\mathbf{x}'}, \\ d\mathbf{x} &\rightarrow \sqrt{\frac{c}{a_2}}d\mathbf{x}', \quad V \rightarrow V'\Big[\frac{c}{a_2}\Big]^{d/2}. \end{aligned}$$

Then the integral becomes

$$\begin{aligned} I &= \int_V \{a_2 s^2(\mathbf{x}) + a_4 s^4(\mathbf{x}) + c[\nabla s(\mathbf{x})]^2\} d\mathbf{x} \\ &= \Big[\frac{c}{a_2}\Big]^{d/2} \int_{V'} \{a_2 s^2(\mathbf{x}') + a_4 s^4(\mathbf{x}') + c[\nabla s(\mathbf{x}')]^2\} d\mathbf{x}'. \end{aligned}$$

Now, changing s to $\sqrt{a_2/a_4}\ s'$, one gets

$$I = \Big[\frac{c}{a_2}\Big]^{d/2} \frac{a_2^2}{a_4} \int_{V'} \{s'^{\,2}(\mathbf{x}') + s'^{\,4}(\mathbf{x}') + [\nabla s'(\mathbf{x}')]^2\} d\mathbf{x}'.$$

Therefore Z can be written as

$$Z = \exp(-a_0 V) J \int Ds' \exp\Big[-\Omega \int_{V'} \{s'^{\,4}(\mathbf{x}') + s'^{\,4}(\mathbf{x}') + [\nabla s'(\mathbf{x}')]^2\} d\mathbf{x}'\Big],$$

where J is the Jacobian associated with the transformation of the order parameter and

$$\Omega = \frac{c^{d/2}}{a_4} [a_2'(T - T_c)]^{2-d/2}.$$

If Ω is large, the integral may be approximated by taking the most dominant contribution. As T approaches T_c from above, Ω becomes large when $d > 4$. Thus the approximation scheme of Landau's theory with Gaussian fluctuations is adequate for $d > 4$.

To provide another argument for the validity of the Landau's approach, consider the expression

$$\chi = \frac{G(0)}{T} = \frac{1}{T} \int_V < s'(\mathbf{x}) s'(0) > d\mathbf{x}.$$

Using the expression obtained in the previous section for χ, when $T < T_c$, one gets

$$\frac{1}{4T a_2'(T_c - T)} = \frac{1}{T} \int_V < s'(\mathbf{x}) s'(0) > d\mathbf{x}.$$

The spin fluctuations have a correlation length ξ' (for $T < T_c$) and for x within ξ', $< s'(\mathbf{x})s'(0) >$ may be taken to be nearly constant. For x larger than ξ', the correlation function is negligible. So, the above relation can be approximated as

$$\frac{1}{4a_2'(T_c - T)} \approx < s'^{\,2} > \xi'^{\,d}.$$

Therefore, the mean square fluctuation in Gaussian approximation is

$$< s'^{\,2} > \approx \frac{\xi'^{\,-d}}{4a_2'(T_c - T)} = \frac{1}{4a_2'}\Big[\frac{c}{2a_2'}\Big]^{-d/2}(T_c - T)^{d/2-1}.$$

It is also known that $< s >=< s'' >$ and

$$< s'' >^2 = \frac{a_2'}{2a_4}(T_c - T).$$

For Landau's theory to be valid, it is necessary that

$$< s >^2 \quad \gg \quad < (s - < s >)^2 > \quad = \quad < s'^{\,2} > .$$

Thus one has the requirement

$$\frac{a_2'}{2a_4}(T_c - T) \quad \gg \quad \frac{1}{4a_2'}\Big[\frac{c}{2a_2'}\Big]^{-d/2}(T_c - T)^{d/2-1},$$

which yields

$$(T_c - T)^{d/2-2} \quad \ll \quad \frac{2a_2'^{\,2}}{a_4}\Big[\frac{2a_2'}{c}\Big]^{d/2}.$$

For this condition to be valid when T is close to T_c, it is required that $d > 4$. Thus once again the conclusion is that the method is consistent only for $d > 4$.

2.5 Ginzburg Criterion

All the previous arguments, which show that fluctuations are significant for dimension $d < 4$, are based on the behavior of thermodynamic quantities predicted by Landau's theory near T_c. When T is far from T_c, it may still be appropriate to use the idea of considering the most probable spin profile and Gaussian fluctuations. In other words, there is a temperature range near T_c where the Landau's approach is inadequate. Ginzburg criterion gives an approximate estimate of this temperature range. The discontinuity in specific heat obtained in Landau's theory is

$$\triangle C = T_c^2 \frac{a_2'^{\,2}}{a_4}.$$

However, addition of Gaussian fluctuations yields a divergence in C given by

$$C \approx C_0 \xi^{4-d},$$

for $d < 4$. Thus the inference is that fluctuations are important for $d < 4$ in some temperature range near T_c. This range can be estimated (in a qualitative manner) by comparing $\triangle C$ and C. Their ratio is

$$\begin{aligned}
\frac{C}{\triangle C} &= \frac{C_0}{\triangle C}\xi^{4-d} \\
&= \frac{C_0}{\triangle C}\Big[\frac{a_2'}{c}\Big]^{d/2-2}(T-T_c)^{d/2-2} \\
&= \Big[\Big[\frac{C_0}{\triangle C}\Big]^{2/(4-d)}\frac{c}{a_2' T_c}\frac{T_c}{(T-T_c)}\Big]^{2-d/2} \\
&= \Big[\frac{\xi_T T_c}{T-T_c}\Big]^{2-d/2},
\end{aligned}$$

where ξ_T is defined as

$$\xi_T = \Big[\frac{C_0}{\triangle C}\Big]^{2/(4-d)}\frac{c}{a_2' T_c}.$$

The dimensionless parameter ξ_T can be estimated from experimental data. Let it be expressed as

$$\xi_T^{(4-d)/2} = \frac{C_0}{\triangle C}\Big[\frac{c}{a_2' T_c}\Big]^{(4-d)/2}.$$

It is known that C_0 is given by

$$\begin{aligned}
C_0 &= \frac{n}{2}\Big[\frac{a_2' T_c}{c}\Big]^2 \frac{I_0}{(2\pi)^d}, \\
I_0 &= \int_0^\infty \frac{d\mathbf{k}}{(1+k^2)^2}.
\end{aligned}$$

Therefore the expression for ξ_T is

$$\begin{aligned}
\xi_T^{2-d/2} &= \frac{nI_0}{2\triangle C}\Big[2\pi\big(\frac{c}{a_2' T_c}\big)^{1/2}\Big]^{-d} \\
&= \frac{nI_0}{2\triangle C}[2\pi\xi_0]^{-d}, \\
\xi_0 &= \Big[\frac{c}{a_2' T_c}\Big]^{1/2}.
\end{aligned}$$

The definition of correlation length is

$$\xi = \Big[\frac{c}{a_2'}\Big]^{1/2}(T-T_c)^{-1/2},$$

and so ξ_0/ξ is given by

$$\frac{\xi_0}{\xi} = \Big[\frac{T - T_c}{T_c}\Big]^{1/2}.$$

Hence, if the correlation length can be measured at any temperature $T(> T_c)$ and further T_c is known, ξ_0 can be estimated for different materials. Then, from specific heat data, one can estimate $\triangle C$ and hence ξ_T. Note that

$$\frac{C}{\triangle C} \geq 1 \; for \; |T - T_c| \in \xi_T T_c.$$

In other words, a divergence in C will be experimentally observed in a temperature range $\xi_T T_c$. Alternatively, one may say that the criterion for the validity of Landau's theory with Gaussian fluctuations is that $|T - T_c|$ is much greater than $\xi_T T_c$.

2.6 Failure of Perturbation Theory

Since some inconsistencies in Landau's theory with Gaussian fluctuations have been noted for $d < 4$, it is necessary to look for a more systematic method. In fact, it is necessary to go beyond the quadratic approximation for the hamiltonian in evaluating the partition function. A straight forward approach would be to account for the quartic terms in the hamiltonian via perturbation theory methods. The hamiltonian can be separated as

$$\frac{H}{T} = \frac{H_0}{T} + \frac{H_1}{T},$$

where H_1/T represents the perturbation terms. Thus the first term H_0/T represents the quadratic terms in H/T. Assuming $n = 1$ (for simplicity) H_0/T and H_1/T are given by

$$\begin{aligned} \frac{H_0}{T} &= \sum_{k \leq \Lambda} (a_2 + ck^2)|s_{\mathbf{k}}|^2, \\ \frac{H_1}{T} &= \frac{a_4}{L^d} \sum_{k_1,k_2,k_3,k_4 \leq \Lambda} s_{\mathbf{k}_1} s_{\mathbf{k}_2} s_{\mathbf{k}_3} s_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4). \end{aligned}$$

The term a_0, which is independent of $s_{\mathbf{k}}$, and h have been omitted. Further, the quartic term has been rewritten with four summations incorporating a delta function. Now consider a_4 to be a small (positive) constant so that H_1/T can be treated as a perturbation. The partition function can be expressed as

$$Z = \int \prod_{k \leq \Lambda} ds_{\mathbf{k}} \exp\Big[\frac{-H_0}{T}\Big]\Big[1 + \frac{H_1}{T} + \frac{1}{2}(\frac{H_1}{T})^2 + \cdots\Big].$$

Now, all the integrals involved in Z are Gaussian integrals and it would appear that one can carry out the perturbation calculations to arbitrary order. Z can also be expressed as

$$Z = Z_0\Big[1 + \frac{< H_1 >}{T} + \frac{1}{2}\frac{< H_1 >^2}{T^2} + \cdots\Big],$$

where Z_0 is the partition function corresponding to H_0/T and $< H_1 > /T$ etc. are averages over the Gaussian distribution $\exp(-H_0/T)$. For the first order term one gets

$$\frac{< H_1 >}{T} = \frac{a_4}{L^d}\sum_{k_1,k_2,k_3,k_4\leq\Lambda} < s_{\mathbf{k}_1}s_{\mathbf{k}_2}s_{\mathbf{k}_3}s_{\mathbf{k}_4} > \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4).$$

The averages over $\exp(-H_0/T)$ can be easily obtained, in fact, it was shown earlier that

$$\begin{aligned} < s_{\mathbf{k}_1}s_{\mathbf{k}_2} > &= \delta(\mathbf{k}_1+\mathbf{k}_2)G_0(k_1), \\ G_0(k) &= \frac{1}{2(a_2+ck^2)}. \end{aligned}$$

Averages of products of four $s_{\mathbf{k}}$ (as occurring in $< H_1 > /T$) can be evaluated with the help of the generating function

$$W(\lambda) = \int\prod_{k\leq\Lambda} ds_{\mathbf{k}} \exp\Big[-\sum_{k\leq\Lambda}(a_2+ck^2)|s_{\mathbf{k}}|^2 + \lambda_{\mathbf{k}}s_{\mathbf{k}}\Big],$$

where $\lambda_{\mathbf{k}}$ is a real parameter. Note that $W(0)$ is same as the partition function. Derivatives of W at $\lambda_{\mathbf{k}} = 0$ yield

$$\begin{aligned} \frac{1}{W(0)}\frac{\partial W(0)}{\partial\lambda_{\mathbf{k}}} &= < s_{\mathbf{k}} >, \\ \frac{1}{W(0)}\frac{\partial^2 W(0)}{\partial\lambda_{\mathbf{k}}\partial\lambda_{-\mathbf{k}}} &= < |s_{\mathbf{k}}|^2 >, \\ \frac{1}{W(0)}\frac{\partial^4 W(0)}{\partial\lambda_{\mathbf{k}_1}\partial\lambda_{\mathbf{k}_2}\partial\lambda_{\mathbf{k}_3}\partial\lambda_{\mathbf{k}_4}} &= < s_{\mathbf{k}_1}s_{\mathbf{k}_2}s_{\mathbf{k}_3}s_{\mathbf{k}_4} > . \end{aligned}$$

Now, as in the calculation of the partition function, which led to Eq.(2.4), rewrite W as

$$W(\lambda) = \int\prod_{k'\leq\Lambda} ds_{\mathbf{k}'}ds_{-\mathbf{k}'} \exp\Big[-\sum_{k'\leq\Lambda} 2(a_2+ck'^{\,2})|s_{\mathbf{k}'}|^2 + \lambda_{\mathbf{k}'}s_{\mathbf{k}'} + \lambda_{-\mathbf{k}'}s_{-\mathbf{k}'}\Big].$$

Since $s_{\mathbf{k}'}$ and $s_{-\mathbf{k}'}$ are complex conjugates, one gets

$$\begin{aligned} W(\lambda) &= 2\int\prod_{k'\leq\Lambda} ds^R_{\mathbf{k}'}ds^I_{\mathbf{k}'} \exp\Big[-\sum_{k'\leq\Lambda} G_0(k')^{-1}(s^{R\,2}_{\mathbf{k}'} + s^{I\,2}_{\mathbf{k}'}) \\ &+ (\lambda_{\mathbf{k}'}+\lambda_{-\mathbf{k}'})s^R_{\mathbf{k}'} + \imath(\lambda_{\mathbf{k}'}-\lambda_{-\mathbf{k}'})s^I_{\mathbf{k}'}\Big]. \end{aligned}$$

Using the result

$$\int_{-\infty}^{\infty} dx \exp(-ax^2 - bx) = \sqrt{\frac{\pi}{a}} \exp\Big[\frac{b^2}{4a}\Big],$$

all the integrals in $W(\lambda)$ can be evaluated to obtain

$$\begin{aligned} W(\lambda) &= 2 \prod_{k' \leq \Lambda} \sqrt{\pi G_0(k')} \exp\Big[\frac{G_0(k')}{4}(\lambda_{\mathbf{k}'} + \lambda_{-\mathbf{k}'})^2\Big] \\ &\times \sqrt{\pi G_0(k')} \exp\Big[-\frac{G_0(k')}{4}(\lambda_{\mathbf{k}'} - \lambda_{-\mathbf{k}'})^2\Big]. \end{aligned}$$

That is

$$W(\lambda) = 2 \prod_{k' \leq \Lambda} [\pi G_0(k')] \exp\Big[\sum_{k' \leq \Lambda} (\lambda_{\mathbf{k}'} \lambda_{-\mathbf{k}'}) G_0(k')\Big].$$

Evaluating the derivatives of W at $\lambda_k = 0$, expressions for $< s_{\mathbf{k}} >$ and $< |s_{\mathbf{k}}|^2 >$ can be easily obtained. By repeated differentiation, one also gets

$$\begin{aligned} < s_{\mathbf{k}_1} s_{\mathbf{k}_2} s_{\mathbf{k}_3} s_{\mathbf{k}_4} > &= \delta(\mathbf{k}_1 + \mathbf{k}_2) G_0(k_2) \delta(\mathbf{k}_3 + \mathbf{k}_4) G_0(k_4) \\ &+ \delta(\mathbf{k}_1 + \mathbf{k}_3) G_0(k_3) \delta(\mathbf{k}_2 + \mathbf{k}_4) G_0(k_4) \\ &+ \delta(\mathbf{k}_2 + \mathbf{k}_3) G_0(k_3) \delta(\mathbf{k}_1 + \mathbf{k}_4) G_0(k_4). \end{aligned}$$

which can be written as

$$\begin{aligned} < s_{\mathbf{k}_1} s_{\mathbf{k}_2} s_{\mathbf{k}_3} s_{\mathbf{k}_4} > &= < s_{\mathbf{k}_1} s_{\mathbf{k}_2} >< s_{\mathbf{k}_3} s_{\mathbf{k}_4} > \\ &+ < s_{\mathbf{k}_1} s_{\mathbf{k}_3} >< s_{\mathbf{k}_2} s_{\mathbf{k}_4} > \\ &+ < s_{\mathbf{k}_1} s_{\mathbf{k}_4} >< s_{\mathbf{k}_2} s_{\mathbf{k}_3} > . \end{aligned}$$

Substitution in $< H_1 > /T$ yields

$$\begin{aligned} \frac{< H_1 >}{T} &= \frac{3a_4}{L^d} \sum_{\{k_i\} \leq \Lambda} \delta(\mathbf{k}_1 + \mathbf{k}_2) G_0(k_2) \\ &\times \delta(\mathbf{k}_3 + \mathbf{k}_4) G_0(k_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \\ &= \frac{3a_4}{L^d} \Big[\sum_{k \leq \Lambda} G_0(k)\Big]^2. \end{aligned}$$

Now consider the limit of large volume $V = L^d$ and replace the summation over k by an integral with a density of states $(L/2\pi)^d$. Thus the first order term in Z is

$$\frac{< H_1 >}{T} = 3a_4 L^d \kappa_d^2 \Big[\int_0^{\Lambda} G_0(k) k^{d-1} dk\Big]^2.$$

To study the behavior of Z (or free energy) near T_c, the limit $T \to T_c$, i.e. $a_2 \to 0$ is to be taken. Then, in $< H_1 > /T$, one gets the integral $\int_0^{\Lambda} k^{d-3} dk$

Table 2.1: Exponents in Gaussian Approximation

β	γ	δ	α	ν	η
$1/2$	1	3	$(4-d)/2$	$1/2$	0

which diverges (at the lower limit) for $d \leq 2$. The second order approximation worked out in a similar way leads to an integral $\int_0^\Lambda k^{d-5} dk$ (for $a_2 = 0$) which diverges for $d \leq 4$. In general, one can show that the n^{th} order approximation diverges for $d \leq 2n$. Thus the direct perturbation theory (where the expansion parameter is a_4) is useless. The divergence arises from modes of small k, i.e. of long wavelength. Thus the conclusion is that long wavelength fluctuations (which are responsible for the divergence of thermodynamic quantities) can not be treated perturbatively.

2.7 Summary

Table 2.1 shows that addition of Gaussian fluctuations to Landau's theory does not alter the exponents β, γ and δ. However, α is found to depend on d for $d < 4$. Landau's theory predicted that $\alpha = 0$ for all d. Further, the Gaussian approximation has provided values for the exponents ν and η. All exponents are independent of model parameters (a_2', a_4, c etc.) and hence the theory shows universality. The spatial dimension ($d = 4$) above which Landau's theory is consistent is known as the upper critical dimension. For n-vector models, two dimension is called the lower critical dimension since there is no sponteneous magnetization for $d \leq 2$. For Ising model, the lower critical dimension is one.

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Chapter 3

Scaling Hypotheses

Scaling hypotheses are attempts to generalize the results of Landau's theory so as to incorporate the experimental facts. In Landau's theory, the average spin variable, which is same as the most probable value, is given by the "equation of state"

$$2s''[a_2 + 2a_4 s''^{\,2}] = h.$$

Throughout this chapter, the order parameter is assumed to have only one component. It was shown that the terms in the square bracket yield the exponent $\beta = 1/2$ when $h = 0$ and

$$a_2 = a_2'(T - T_c).$$

So, to incorporate the correct β, the equation of state can be modified as

$$2s''[a_2 + 2a_4 s''^{\,1/\beta}] = h.$$

To develop scaling hypotheses, the equations of Landau's theory are rewritten with the general exponents as shown in the above example. Substituting for a_2 and dividing by $|T - T_c|^{3/2}$ one gets

$$2s''\Big[\pm \frac{a_2'}{|T - T_c|^{1/2}} + \frac{2a_4 s''^{\,1/\beta}}{|T - T_c|^{3/2}}\Big] = \frac{h}{|T - T_c|^{3/2}},$$

where + sign is for $T > T_c$ and − sign is for $T < T_c$. This equation can be rewritten as

$$\frac{s''}{B|T - T_c|^{1/2}}\Big[\pm 1 + \frac{1}{B^2}\Big(\frac{s''}{|T - T_c|^{\beta}}\Big)^{1/\beta}\Big] = \frac{Dh}{|T - T_c|^{3/2}},$$

where

$$\begin{aligned} B &= \sqrt{a_2'/(2a_4)}, \\ D &= \frac{1}{2a_2' B}. \end{aligned}$$

Again, replacing 1/2 with β, B^2 with $B^{1/\beta}$ and introducing $\Delta = 3/2$ one finds

$$\frac{s''}{B|T-T_c|^\beta}\Big[\pm 1 + \Big(\frac{s''}{B|T-T_c|^\beta}\Big)^{1/\beta}\Big] = \frac{Dh}{|T-T_c|^\Delta}.$$

The parameter Δ will be related to the susceptibility exponent γ later. Since

$$s'' \sim |T-T_c|^\beta, \ for\ T \sim T_c,$$

the quantity $s''/(B|T-T_c|^\beta)$ is a scaled spin variable. Similarly, $Dh/|T-T_c|^\Delta$ is the magnetic field scaled by a specific power of $|T-T_c|$. An important observation is that the equation of state can be written in terms of two scaled variables $s''/(B|T-T_c|^\beta)$ and $Dh/|T-T_c|^\Delta$ rather than with s'', T and h as three independent variables. Though this observation is from Landau's theory, Widom hypothesized that it is true in actual systems near the critical point, i.e. when $T \approx T_c$ and $h \approx 0$.

3.1 Scaling Hypothesis for Order Parameter

Widom's scaling hypothesis for order parameter is

$$\frac{s''}{B|T-T_c|^\beta} = W_\pm\Big[\frac{Dh}{|T-T_c|^\Delta}\Big],$$

where W_+ (for $T > T_c$) and W_- (for $T < T_c$) are two functions of a single variable. The validity of this hypothesis is to be checked against experimental observations. According to Landau's theory, $\beta = 1/2$, $\Delta = 3/2$ and $W_\pm(x)$ are universal functions. Only the amplitudes B and D depend on the details (a_2' and a_4) of the system. Before studying the predictions of Widom's hypothesis, a relation connecting γ and Δ can be obtained. For $T > T_c$, the susceptibility is

$$\chi = (\frac{\partial s''}{\partial h})_{h=0} \quad \approx BD(T-T_c)^{\beta-\Delta}W_+'(0),$$

where $W_+'(x)$ is the derivative of $W_+(x)$. Thus, the parameter Δ, introduced in the scaling hypothesis, is to be related to γ as

$$\Delta = \beta + \gamma,$$

since $\chi \approx (T-T_c)^{-\gamma}$.

A prediction of the scaling hypothesis is Widom's scaling law which relates the exponents δ (defined for $T = T_c$) and β and γ (defined for $T \neq T_c$). To get this relation, assume that $W_+(x)$ varies as some power of x as $x \to \infty$. That is,

$$W_+(x) \approx a_+ x^\lambda \ as\ x \to \infty.$$

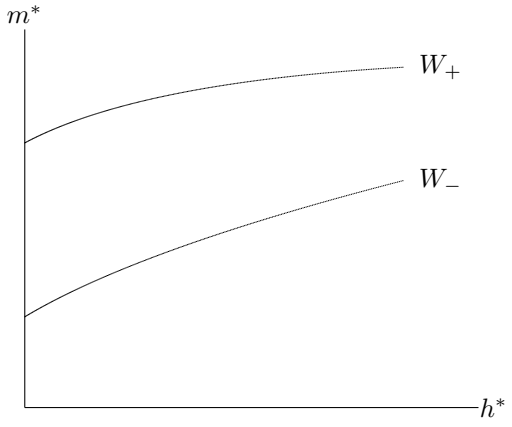

Figure 3.1: Scaled Magnetization Vs Scaled Field.

Then, for $T \approx T_c$, Widom's hypothesis shows that

$$\frac{s''}{B|T-T_c|^{\beta}} \approx a_+\Big[\frac{Dh}{|T-T_c|^{\Delta}}\Big]^{\lambda}.$$

That is

$$s'' \approx Ba_+D^{\lambda}h^{\lambda}|T-T_c|^{\beta-\Delta\lambda}.$$

For $T \approx T_c$, the temperature dependence of s'' should vanish since it is known that $s'' \approx h^{1/\delta}$. Thus, it is required that

$$\lambda = \frac{\beta}{\Delta}.$$

Then one obtains

$$s'' \approx Ba_+D^{\lambda}h^{\lambda},$$

and hence $\lambda = 1/\delta = \beta/\Delta$. Thus δ is given by

$$\delta = \frac{\Delta}{\beta} = \frac{\beta+\gamma}{\beta}, \tag{3.1}$$

which is Widom's scaling law. The experimental validity of this relation will be discussed later. There are a few experimental facts which support the scaling hypothesis.

(*i*) The measured equation of state data (i.e. relation connecting m, T and h) show that when $m^* = m/|T-T_c|^{\beta}$ is plotted against $h^* = h/|T-T_c|^{\beta\delta}$, one gets just two functions (Figure 3.1) for all temperatures. Thus all the data for different temperatures fall on these two curves, one (W_+) for $T > T_c$ and the other (W_-) for $T < T_c$.

(*ii*) The equation of state for different materials are found to fall on identical curves (within experimental errors) when proper amplitude factors B and D

are introduced in the scaled variables. These experimental facts confirm the existence of universal functions $W_+(x)$ and $W_-(x)$ in the scaling hypothesis.

(*iii*) Furthermore, similar equation of state data for different fluids also fall on the same curves (of magnets) and thus establish the universality of the functions W_+ and W_-.

(*iv*) The scaling hypothesis shows that χ should vary as

$$\chi \approx \begin{cases} BD(T-T_c)^{-\gamma}W'_+(0)\ , & T > T_c \\ BD(T_c-T)^{-\gamma}W'_-(0)\ , & T < T_c \end{cases} .$$

Thus, if the amplitudes are denoted as c_+ and c_- for $T > T_c$ and $T < T_c$ respectively, then their ratio is

$$\frac{c_+}{c_-} = \frac{W'_+(0)}{W'_-(0)}.$$

If the scaling hypothesis is correct, this should be a universal constant (for given d and n). Landau's theory gives $c_+/c_- = 2$, while numerical results show that $c_+/c_- \approx 5.03$ for 3-D Ising model. The exact 2-D Ising model calculations show that $c_+/c_- = 37.694$. In addition to exposing the inadequacy of Landau's theory, these results show that the scaling functions $W_\pm(x)$ will depend on the spatial dimension d and (most likely) order parameter dimension n.

3.2 Scaling Hypothesis for Free Energy Density

As was done for the order parameter, the scaling form for free energy density also can be motivated using the Landau's theory expression. The hamiltonian (with $n = 1$) in the Gaussian approximation is

$$\frac{H}{T} \;=\; \frac{H(s'')}{T} + \sum_{k\le\Lambda}(a_2 + 6a_4 s''^{\,2} + ck^2)|s_{\mathbf{k}}|^2. \tag{3.2}$$

Therefore, following the calculations of Chapter 2, the free energy density can be written as

$$F(T,h) \;=\; F_L - \frac{T}{2(2\pi)^d}\int_0^\Lambda \ln\Big[\frac{\pi}{a_2 + 6a_4 s''^{\,2} + ck^2}\Big]d\mathbf{k}. \tag{3.3}$$

The divergence of specific heat, within this approximation, arises from the second part of F and so it is termed as the "singular" part, F_s. Note that F_s can be written as a function of $a_2 \sim T - T_c$ and $s''^{\,2}/a_2$. The scaling hypothesis for s'' was motivated by the replacement

$$\frac{s''^{\,2}}{a_2} \sim \frac{s''^{\,2}}{T-T_c} \quad\to\quad [s''(T-T_c)^{-\beta}]^{1/\beta}.$$

According to the Widom's hypothesis, this term is a function of $h/|T-T_c|^{\Delta}$. Therefore, the singular part F_s may be written as

$$F_s \approx X_{\pm}\Big[|T-T_c|, \frac{h}{|T-T_c|^{\Delta}}\Big],$$

where $X_{\pm}$ are some functions of two variables. This constitutes the scaling hypothesis for free energy density. Now, it is known that

$$C = -T\frac{\partial^2 F}{\partial T^2} \sim |T-T_c|^{-\alpha},$$

when $h = 0$. Hence, the singular part of free energy density should be of the form

$$F_s \approx |T-T_c|^{2-\alpha} Y_{\pm}\Big[\frac{h}{|T-T_c|^{\Delta}}\Big].$$

The same result can also be obtained with a slightly different argument. The magnetization is given by

$$m = s'' = -\frac{\partial F}{\partial h}.$$

Therefore, the non-analytic part of s'' is $\partial F_s/\partial h$. Thus Widom's hypothesis is equivalent to the ansatz

$$F_s \approx |T-T_c|^{x} Y_{\pm}\Big[\frac{h}{|T-T_c|^{\Delta}}\Big],$$

where x is a parameter to be determined. Once again, the dependence of specific heat on temperature shows that $x = 2-\alpha$ and thus one gets the same form for F_s. Using the definition $s'' = -\partial F_s/\partial h$, the exponents α, β and Δ can be related. Differentiation of F_s yields

$$s'' = -\frac{\partial F_s}{\partial h} \;=\; -\,|T-T_c|^{2-\alpha-\Delta}\; Y'_{\pm}\Big[\frac{h}{|T-T_c|^{\Delta}}\Big].$$

Note that the functions $-Y'_{\pm}(x)$ are same as $W_{\pm}(x)$ introduced in the scaling form of s''. Since

$$s''(h=0) \sim\; |T-T_c|^{\beta},$$

one gets the relation

$$\beta = 2-\alpha-\Delta.$$

Now, use of the relation, $\Delta = \beta+\gamma$, yields the Essam-Fisher scaling law

$$\alpha + 2\beta + \gamma \;=\; 2. \tag{3.4}$$

Eliminating γ with the help of Widom's scaling law, this relation can also be written as

$$\alpha + \beta(1+\delta d) \;=\; 2. \tag{3.5}$$

Thus, among the four exponents $\alpha,\ \beta,\ \gamma$ and Δ, there are two independent relations and hence if any two exponents are known, the other two can be computed. These two relations, predicted by the scaling hypotheses, were originally derived as inequalities, the Rushbroke's inequality

$$\alpha + \beta(1+\delta) \geq 2 \ ,$$

and the Griffith's inequality

$$\alpha + 2\beta + \gamma \geq 2 \ ,$$

from thermodynamic considerations. Successive derivatives of F_s w.r.t h (at $h = 0$) have temperature exponents differing by Δ. So Δ is usually known as the gap exponent. It may be noted that all the three relations can be obtained from the scaling ansatz for free energy density.

3.3 Scaling Hypothesis for Correlation Function

The hamiltonian of the Gaussian approximation, given in Eq.(3.2), shows that the Fourier transform of the correlation function is

$$G(T,h,k) = \frac{1}{2}\frac{1}{a_2 + 6a_4 s''^{\ 2} + ck^2}.$$

This expression can be rewritten as

$$G(T,h,k) = \frac{1}{2c}\frac{c}{a_2}\frac{a_2}{a_2 + 6a_4 s''^{\ 2} + ck^2}.$$

Using the definition of correlation length, $\xi^2 = c/a_2$, one finds that

$$G(T,h,k) = \frac{1}{2c}\frac{1}{k^2}\frac{\xi^2 k^2}{1 + 6a_4 s''^{\ 2}/a_2 + \xi^2 k^2}.$$

According to the scaling hypothesis, $s''^{\ 2}/a_2$ is a function of $h/|T-T_c|^{\Delta}$. Then, incorporating the exponent η (which is zero in the Gaussian approximation), the scaling ansatz for correlation function is found to be

$$G(T,h,k) = k^{-2+\eta} D_{\pm}\Big[\xi k, \frac{h}{|T-T_c|^{\Delta}}\Big],$$

where $D_{\pm}$ depend on two scaled variables. The scaled field $h/|T-T_c|^{\Delta}$ appearing in G is the same as that in s'' and F_s.

The correlation length ξ, which diverges near T_c, is one of the characteristic lengths of the system. Other lengths like the atomic spacing and range of interaction are negligible when compared to ξ near T_c. Therefore, as done

above, one may assume that the important length scale of systems near the critical point is ξ and incorporate that in the scaling form for the correlation function. This assumption is in accordance with the universality observed in critical behavior - systems differing in small length scales are all characterized by the same exponents. Since

$$\xi \sim |T - T_c|^{-\nu},$$

G can also be written as

$$G(T, h, k) = k^{-2+\eta} D_{\pm}\Big[\frac{k}{|T - T_c|^{\nu}}, \frac{h}{|T - T_c|^{\Delta}}\Big].$$

Now, note that $\chi = G(k = 0)/T$ and when $h = 0$,

$$\chi \sim |T - T_c|^{-\gamma}.$$

Therefore, χ can be expressed as

$$\chi \sim \lim_{k \to 0} k^{-2+\eta} D_{\pm}\Big[k|T - T_c|^{-\nu}, 0\Big].$$

For the r.h.s to have a finite limit as $k \to 0$, one should have

$$D_{\pm}(x, 0) \to x^{2-\eta} \; as \; x \to 0.$$

Hence, $D_{\pm}(k|T - T_c|^{-\nu}, 0)$ should vary as $(k|T - T_c|^{-\nu})^{2-\eta}$ as $k \to 0$. This implies that

$$\chi \sim |T - T_c|^{-\nu(2-\eta)}.$$

Therefore, one gets a new scaling law

$$\gamma = \nu(2 - \eta), \tag{3.6}$$

connecting γ, ν and η. The three independent scaling laws obtained so far show that, out of the six exponents, only three are independent.

3.4 Hyperscaling Law

All the three scaling laws obtained do not involve the spatial dimension d. The hyperscaling law relates d, ν and α. When $h = 0$ and $T > T_c$, the scaling form obtained for free energy density yields

$$F_s \approx |T - T_c|^{2-\alpha} Y_+(0).$$

The Gaussian approximation of Eq.(3.3), with $s'' = 0$ (since $h = 0$ and $T > T_c$) gives

$$F_s \approx \frac{T\kappa_d}{2} \int_0^{\Lambda} \ln(a_2 + ck^2) k^{d-1} dk.$$

Integration by parts yields

$$F_s \approx \frac{T\kappa_d}{2d}\Big[\Lambda^\delta \ln(a_2 + c\Lambda^2) - 2c\int_0^\Lambda \frac{k^{d+1}dk}{a_2 + ck^2}\Big].$$

Since the first term does not yield a divergence in the derivatives, the singular part of F is

$$F_s \approx -\frac{T}{2}\kappa_d \frac{2c}{d}\int_0^\Lambda \frac{k^{d+1}dk}{a_2 + ck^2}.$$

Changing the variable to $k' = k\xi$, where $\xi^2 = c/a_2$, one gets

$$F_s \approx \frac{1}{\xi^d}\int_0^{\xi\Lambda} \frac{k'^{\,d+1}dk'}{1 + k'^2}.$$

Rewriting the integrand as

$$\frac{k'^{\,d+1}}{1 + k'^{\,2}} = k'^{\,d-1} - \frac{k'^{\,d-1}}{1 + k'^{\,2}},$$

F_s becomes

$$F_s \approx \frac{1}{\xi^d}\Big[\frac{\xi^d\Lambda^d}{d} - \int_0^{\xi\Lambda} \frac{k'^{\,d-1}dk'}{1 + k'^{\,2}}\Big].$$

The integral on the r.h.s has already been evaluated in Chapter 2. Thus for large ξ one finds

$$F_s \approx f_0 + \frac{f_1}{\xi^2} + \frac{f_2}{\xi^4} + \frac{f_3}{\xi^d},$$

where f_0 etc. are constants. The temperature dependent contribution to specific heat arises only from the last term since $\xi^{-2} \sim |T - T_c|$. Hence the singular part of free energy density varies as $F_s \sim \xi^{-d}$. Now, the experimental fact that $\xi \sim |T - T_c|^{-\nu}$ can be used to conclude that

$$F_s \sim |T - T_c|^{\nu d}.$$

Comparing with the scaling form (when $h = 0$), one finds that $2 - \alpha = \nu d$ or a new scaling law

$$\alpha = 2 - \nu d. \tag{3.7}$$

This relation involving the spatial dimension is known as the hyperscaling law.

3.5 Scaling Laws from Scale Transformations

In this section, the four scaling laws are rederived from scale transformations. The spatial length scale characterizing systems near the critical point is the

correlation length. Further, its temperature dependence is $\xi \sim |T - T_c|^{-\nu}$ where ν is the correlation length exponent. If the dependence of various physical quantities on length scales can be determined, their temperature dependence may be obtained by assuming that ξ is the only important length scale in the system. Of course, this assumption is meaningful only near T_c. First of all, consider scale transformations to determine the dependence of physical quantities on length scales.

If the unit of length is changed by a factor q, then the spatial interval Δx changes to $\Delta x' = \Delta x/q$. Therefore the scale dimension of Δx is -1. Thus, the scale dimension of wave vector $\mathbf{k}$ is $+1$. In general, if any quantity A changes to $A' = Aq^l$, then its scale dimension is $d(A) = l$. Thus the scale dimension of volume is $d(V) = -d$. The scale dimension of ξ is, of course, -1. According to the scaling hypothesis, the correlation function $G(k)$ for $h = 0$ is of the form

$$G(k) \sim \xi^{2-\eta} Q(\xi k),$$

where $Q(x)$ is related to D_+ (introduced earlier) as

$$Q(x) = x^{\eta-2} D_+(x, 0).$$

This form for $G(k)$ is based on the assumption that it can be expressed in terms of the scaled wave vector, and ξ is the only important length scale. This relation then shows that the scale dimension of $G(k)$ is $d(G) = \eta - 2$. Now, from definition

$$G(k) \ \sim \ < (spin\ density)^2 > V.$$

Hence the scale dimension of spin density s is

$$d(s) = \frac{1}{2}[d + d(G)] = \frac{1}{2}(d + \eta - 2).$$

The total free energy is independent of change in length scale, but the free energy density (F) is proportional to V^{-1} and hence $d(F) = d$. Now, magnetization is

$$m =< spin\ density > \ = -\partial F/\partial h.$$

Hence the scale dimension of h is

$$d(h) = d - d(s) = \frac{1}{2}(d - \eta + 2).$$

It may be surprising that an externally applied field is found to be altered by a change of the length unit for the system. For consistency of the various thermodynamic relations, the field has to be modified, as one is accepting the scaling form of $G(k)$. Finally, from the definition of Fourier amplitude,

$$s(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_k s_{\mathbf{k}} \exp(\imath \mathbf{k} \cdot \mathbf{x}),$$

Table 3.1: Scale Dimensions.

quantity	*dimension*
Δx	-1
k	$+1$
ξ	-1
$G(k)$	$\eta - 2$
$s(\mathbf{x})\ , m$	$(d + \eta - 2)/2$
$s_{\mathbf{k}}$	$(\eta - 2)/2$
F	d
V	$-d$
h	$(d - \eta + 2)/2$

one gets $d(s_{\mathbf{k}}) = (\eta - 2)/2$. These results are summarized in Table 3.1.

The hyperscaling law can be obtained in the following way. The dependence of F on the scale parameter q is as q^d. Since $\xi \sim q^{-1}$, the dependence of F on ξ should be as ξ^{-d} so that $F^{1/d}$ and ξ^{-1} have the same dimension. Now,

$$C = -T\frac{\partial^2 F}{\partial T^2} \quad \sim |T - T_c|^{\nu d - 2},$$

since $\xi \sim |T - T_c|^{-\nu}$. Since C varies as $|T - T_c|^{-\alpha}$, one gets the hyperscaling law,

$$\alpha = 2 - \nu d.$$

Since $m \sim q^{d(s)}$, m is proportional to $|T - T_c|^{\nu d(s)}$. Thus one finds that $\beta = \nu d(s)$ since the exponent should be β. That is,

$$\beta = \frac{\nu}{2}(d + \eta - 2). \tag{3.8}$$

Again, $m \sim q^{d(s)}$ and $h \sim q^{d(h)}$, which mean that $m \sim h^{d(s)/d(h)}$. Since $m \sim h^{1/\delta}$, one gets $\delta^{-1} = d(s)/d(h)$, and hence

$$\delta = \frac{d - \eta + 2}{d + \eta - 2}. \tag{3.9}$$

Further, note that

$$\frac{m}{h} \ \sim \ q^{d(s)-d(h)} \ \sim \ \xi^{d(h)-d(s)} \ \sim \ |T - T_c|^{\nu\{d(s)-d(h)\}}.$$

Therefore one finds that

$$\chi = \frac{\partial m}{\partial h} \sim |T - T_c|^{\nu\{d(s)-d(h)\}}.$$

Then, the relation $\chi \sim |T - T_c|^{-\gamma}$ yields

$$\gamma = \nu(2 - \eta).$$

It is important to summarize the points emerging from the above derivation of scaling laws.

(*i*) All the four relations have come about by matching the scale dimensions of various quantities. But the scale dimensions $d(G), d(s), d(s_{\mathbf{k}})$ and $d(h)$ were obtained using the scaling ansatz for the correlation function which is based on the assumption that the correlation length is the only important length scale near the critical point.

(*ii*) Using the hyperscaling relation, Eqs.(3.8) and (3.9) can be rewritten as the Widom's scaling law in Eq.(3.1) and the Essam-Fisher scaling law in Eq.(3.4). Therefore the basic relations may be collected together as

$$\begin{aligned} \delta &= (\beta + \gamma)/\beta, \\ \alpha &+ 2\beta + \gamma = 2, \\ \gamma &= \nu(2 - \eta), \\ \alpha &= 2 - \nu\, d. \end{aligned}$$

(*iii*) Thus, there are four independent relations among the six exponents, and hence only two exponents are independent. Note that the two definitions, one for ξ and the other for $G(k)$, were used in the derivation of scaling laws.

(*iv*) The hyperscaling law is satisfied by the exponents in Gaussian approximation ($\alpha = 2 - d/2$ for $d < 4$, $\alpha = 0$ for $d \geq 4$ and $\nu = 1/2$) only for $d < 4$. Since the Gaussian approximation is expected to be correct for $d \geq 4$, it should be suspected that the hyperscaling law is valid only for $d < 4$.

(*v*) The 2-D Ising model exponents, $\alpha = 0,\ \beta = 1/8,\ \nu = 1,\ \eta = 1/4,\ \gamma = 7/4$ and $\Delta = 15$, satisfy the scaling relations.

(*vi*) For isotropic ferromagnets ($n = 3$), putting experimental values $\gamma = 1.33$ and $\eta = 0.07$, the scaling relations yield $\nu = 0.69,\ \alpha = -0.07,\ \beta = 0.37$ and $\delta = 4.6$. For liquid-gas transition ($n = 1$), experimental values $\gamma = 1.20$ and $\eta = 0.11$ and the scaling relations yield $\nu = 0.64,\ \alpha = 0.08,\ \beta = 0.36$ and $\delta = 4.4$. These 'calculated' results are in agreement with the measured values within 10% accuracy.

3.6 Kadanoff Transformation and Scaling

Kadanoff has developed a different set of arguments which lead to the scaling ansatz for free energy density and correlation function and hence the scaling

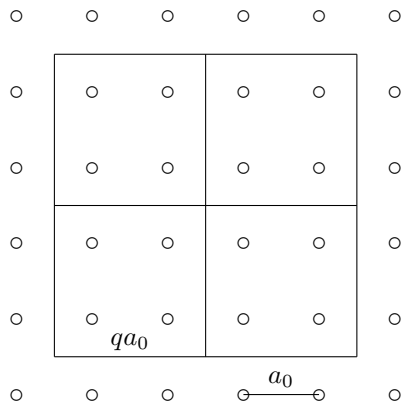

Figure 3.2: Ising Lattice and Cells.

laws. Consider the Ising model and imagine that the lattice is divided into cells as shown in Figure 3.2. The side length of a cell is qa_0 where a_0 is the lattice spacing. The parameter q is chosen such that $a_0 \ll qa_0 \ll \xi$, the correlation length. Thus there are a number of lattice points in each cell.

If the interactions between the cells are turned off, the correlation length will be less than qa_0. Since $\xi \gg qa_0$ in the actual case, it can be concluded that the interactions between the cells are very important. Experimental observation of universality indicates that the cooperative behavior of a system close to the critical point is insensitive to features over length scales smaller than ξ. The cell size is one of such insensitive parameters.

An average spin parameter μ_c, where c denotes the cell index, may be associated with each cell. For $q = 1$, one should have $\mu_c = s_i$. It can be expected that, for $\xi \gg qa_0$, most of the spins in a particular cell are in the same direction. Further, the average spin parameters $\{\mu_c\}$ should be such that the interaction among them and with an external field yield the long range correlations existing in the original system. Thus, the attempt is to average out the short distance variations of the spins s_i and make an Ising model with average spin parameters $\{\mu_c\}$ so that the new model has the same characteristics over long length scales. It is a hypothesis that a new model, satisfying these requirement, can be constructed.

The new Ising model, which has a spacing qa_0, is supposed to have the same correlation length. If the correlation lengths are measured in units of lattice spacing, the correlation length of the new model is much smaller than that of the old model. Note that the correlation length is always to be compared to a basic length scale, such as lattice spacing, to decide whether the system is near the critical point or not. Because of the small correlation length, an

independent view of the new system shows that it is away from the critical point than the original system. This means that the temperature and field parameters of the new model are different from those of the old model. If the critical temperatures of the two models are same, then the parameters

$$\epsilon = (T - T_c)/T_c \ , \quad h = h_{ex}/T \ ,$$

will have different values ϵ' and h' for the new model. The parameters ϵ' and h' should describe, as said earlier, the interaction between the cells and with an external field. Note that ϵ' contains T_c and hence the coupling constant of the Ising model. If the external field is absent in the original model, it should be so in the new model also. Further, h' should depend on the cell size parameter (q) in such a way that $h' = h$ when $q = 1$. Therefore, one may assume that $h' = q^x h$ where x is some number. According to the discussion on scale transformations, x is the scale dimension of the field. In a similar way, it may be assumed that when original system is critical ($\epsilon = 0$), the new system also is critical ($\epsilon' = 0$). Then, as in the case of h, one may assume the relation $\epsilon' = q^y \epsilon$ where y is some other number. These assumptions yield the scaling ansatz for free energy density and correlation function.

3.6.1 Relation between Cell and Site Spins

To find a relation connecting μ_c and s_c, consider the change in energy ΔE due to a change Δh in the field. Here, s_c is the site spin for some i (site index) belonging to the c^{th} cell. ΔE is given by

$$\Delta E = \sum_i \Delta h s_i.$$

This change in energy should be the same in the new model also. That is

$$\Delta E = \sum_c \Delta h' \mu_c.$$

Assuming that the spins in a particular cell are in the same direction, one gets

$$\Delta E = \sum_c \Delta h \sum_{i \in c} s_i \ = \ \sum_c \Delta h q^d s_c,$$

where s_c, the site spin for any i belonging to cell c, takes values ± 1. Substituting for Δh, one gets

$$\Delta E = \sum_c \Delta h' q^{-x} q^d s_c,$$

which implies that

$$\mu_c = q^{d-x} s_c.$$

3.6.2 Scaling of Free Energy Density

Let the singular part of free energy density of the original model be denoted as $F_s(\epsilon, h)$. Since the new model is again an Ising model, the singular part of free energy density will be the same function $F_s(\epsilon', h')$ of the variables ϵ' and h'. There are q^d original spins in each of the cells of the new model. Therefore, one gets the relation

$$q^d F_s(\epsilon, h) = F_s(\epsilon', h').$$

Substitution of ϵ' and h' leads to a functional equation

$$F_s(\epsilon, h) = q^{-d} F_s(\epsilon q^y, h q^x).$$

According to Kadanoff's assumptions, the parameter q (with the condition $a_0 \ll q a_0 \ll \xi$) is arbitrary. That is, the functional equation must be satisfied for all q satisfying this condition. This is possible only if $F_s(\epsilon, h)$ is of the form

$$F_s(\epsilon, h) = |\epsilon|^{d/y} Y\Big(\frac{h}{|\epsilon|^{x/y}}\Big).$$

where Y is a suitable function of the scaled variable $h/|\epsilon|^{x/y}$. By direct substitution, it can be easily verified that this form for F_s satisfies the functional equation. With the identifications, $2 - \alpha = d/y$ and $\Delta = x/y$, the above expression is found to be the same as the scaling form for free energy density.

3.6.3 Temperature Dependence of ξ

The original model and the cell model should have the same correlation length. That is, $\xi(\epsilon, h) = \xi(\epsilon', h')$. Since both models are Ising models, the functional dependence of ξ on temperature and field variables in both models is same. Introducing the lattice spacing a_0, the equality can be expressed as

$$\frac{\xi(\epsilon, h)}{a_0} = q \frac{\xi(\epsilon', h')}{q a_0}.$$

If ξ^+ denotes the correlation length in units of lattice spacing, the functional equation becomes

$$\xi^+(\epsilon, h) = q \xi^+(\epsilon q^y, h q^x),$$

where the expressions for ϵ' and h' have been used. This fact, that the correlation length (in units of lattice spacing) is smaller for the new model by a factor q, was stated earlier. Since q is arbitrary, the solution of the functional equation is of the form

$$\xi^+(\epsilon, h) = |\epsilon|^{-1/y} f\Big(\frac{h}{|\epsilon|^{x/y}}\Big),$$

where f is a function of the scaled field variable. For $h = 0$, one gets

$$\xi^+(\epsilon, 0) = |\epsilon|^{-1/y} f(0),$$

which is the experimentally observed divergence of correlation length if $\nu = y^{-1}$. Combining with the relation, $2 - \alpha = d/y$ obtained earlier, one gets the hyperscaling law $\alpha = 2 - \nu d$.

3.6.4 Scaling of Correlation Function

The spatial correlation function of the original Ising model is

$$G(r, \epsilon, h) =< s_i \; s_{i+r} >,$$

Let all distances be measured in units of the lattice spacing a_0. As in the case of correlation length, it is necessary to consider distances in natural units to obtain a functional relation for the correlation function. The cell spins associated with s_i and s_{i+r} are μ_c and $\mu_{c+r/q}$. Their separation in units of the cell lattice spacing is $r' = r/q$. The correlation function $< \mu_c \; \mu_{c+r/q} >$ of the new model is the same function G of the variables r', ϵ' and h'. Using the relation $\mu_c = q^{d-x} s_c$, one gets

$$\begin{aligned} G(r, \epsilon, h)(q^{d-x})^2 &= < s_i \; s_{i+r} > (q^{d-x})^2 \\ &= < \mu_x \; \mu_{x+r/q} > \\ &= G(r/q, \epsilon', h'). \end{aligned}$$

Again, substitution of ϵ' and h' yields the functional equation

$$G(r, \epsilon, h) = q^{-2(d-x)} G(r/q, \epsilon q^y, h q^x).$$

Solution of this functional equation is

$$G(r, \epsilon, h) = |\epsilon|^{2(d-x)/y} \; g\Big(r|\epsilon|^{1/y}, \frac{h}{|\epsilon|^{x/y}}\Big).$$

The Fourier transform of the correlation functions are, then, related as

$$\begin{aligned} G(k, \epsilon, h) &= |\epsilon|^{2(d-x)/y} \int \exp(\imath \mathbf{k} \cdot \mathbf{r}) g\Big[r|\epsilon|^{1/y}, \frac{h}{|\epsilon|^{x/y}}\Big] d\mathbf{r} \\ &= |\epsilon|^{2(d-x)/y - d/y} \int \exp\Big[\imath \mathbf{k} \cdot \mathbf{z} |\epsilon|^{-1/y}\Big] g\Big[z, \frac{h}{|\epsilon|^{x/y}}\Big] d\mathbf{z} \\ &= |\epsilon|^{(d-2x)/y} g(k|\epsilon|^{-1/y}, \frac{h}{|\epsilon|^{x/y}}). \end{aligned}$$

With $y^{-1} = \nu$, and hence $|\epsilon|^{-1/y} \sim |T - T_c|^{-\nu} \sim \xi$, one gets

$$G(k, \epsilon, h) = k^{(d-2x)} D\Big(k\xi, \frac{h}{|T - T_c|^{\Delta}}\Big),$$

where

$$D\Big(k\xi, \frac{h}{|T-T_c|^{\Delta}}\Big) = [\frac{k}{\xi}]^{2x-d}\; g\Big(k\xi, \frac{h}{|\epsilon|^{x/y}}\Big),$$

and

$$\Delta = \frac{x}{y}\ , \quad x = \frac{1}{2}(d+2-\eta).$$

Thus the scaling form of correlation function can also be motivated using Kadanoff's ideas. Note that the expressions for x and $d-x=(d-2+\eta)/2$ match with the scale dimensions of h and magnetization (or spin density) obtained earlier. The main assumptions of Kadanoff's derivation of the scaling behavior are the following.

(i) Since correlation length is very large near the critical point, one may define a new Ising model with cell averaged spin variables where the cell size q satisfies the conditions $a_0 \ll qa_0 \ll \xi$.

(ii) The new spin variables for the cells are to be related to the old variables with the requirement that the interactions of the new variables among themselves and with an external field reproduce the long range correlations in the original model.

(iii) Parameters of the models are related by power laws.

3.7 Cell Hamiltonian and Kadanoff Transformation

Since the cell averaging procedure, introduced by Kadanoff, explains the scaling behavior, a rigorous way to obtain the cell hamiltonian is discussed below. The cell spin variable is defined as the average of the spins belonging to a cell, i.e. the sum of spin values in the cell divided by q^d. For describing the spin variations over distances larger than qa_0, these cell averaged variables are expected to be adequate. However, they can not describe spin variations over distances smaller than qa_0. This feature is similar to that in an Ising model where the cut-off length for spin variations is the lattice spacing a_0. The Landau-Ginzburg model accounts for spin variations over distances larger than a cut-off value $b = 2\pi/\Lambda$.

Now, recall that if $P(q_1, q_2)$ is the joint distribution of two random variables, Q_1 and Q_2, then the distribution of $Q = (Q_1 + Q_2)/2$ is given by

$$P(q) = \int P(q_1,q_2)\delta\Big[q - \frac{1}{2}(q1+q2)\Big]dq_1dq_2 \;\; = \;\; <\delta\Big[q-\frac{1}{2}(q1+q2)\Big]> .$$

The cell spin is defined as

$$s_c = q^{-d}\sum_{i\in c} s_i,$$

The symbol $\sum_{i\in c}$ indicates summation over all lattice points i belonging to the cell c. The cell parameter μ_c of previous section is proportional to s_c. Thus the probability distribution of the cell variables is

$$\begin{aligned} P'[s] &= <\prod_c \delta\Big[s_c - q^{-d}\sum_{i\in c} s_i\Big]>, \\ &= \frac{1}{Z}\int \exp\Big[-\frac{H[s]}{T}\Big]\prod_c \delta\Big[s_c - q^{-d}\sum_{i\in c} s_i\Big]\prod_i ds_i, \end{aligned}$$

where $H[s]$ and Z are respectively the hamiltonian and partition function of the Ising model. The hamiltonian $H'[s]$ and partition function Z' of the cell model are related as

$$P'[s] = \frac{1}{Z'}\exp\Big[-\frac{H'[s]}{T}\Big].$$

The two probability distributions, $P[s]$ and $P'[s]$, or the hamiltonians $H[s]$ and $H'[s]$, are equivalent as far as spin variations over distances greater than qa_0 are concerned. Obtaining $H'[s]$ from a given $H[s]$ is called a Kadanoff transformation K_q. It may symbolically be written as

$$H'[s] = K_q H[s],$$

where q is the cell size parameter. Obviously, K_1 (obtained with $q = 1$) is the identity transformation. If another Kadanoff transformation $K_{q'}$ is performed, one gets

$$H''[s] = K_{q'}H'[s] = K_{q'}K_q H[s].$$

Two transformations lead to a cell size parameter $q'q$. Therefore, $H''[s]$ may also be written as

$$H''[s] = K_{qq'}H[s].$$

Thus the transformations $\{K_q\}$ have the property

$$K_{q'}K_q = K_{qq'}.$$

Kadanoff transformation can not produce any singular behavior, in the thermodynamic quantities, since the long wave length variations of spins, i.e. variations on a scale larger than qa_0, are unaltered by it.

3.8 Finite Size Scaling

The scaling hypotheses introduced so far, for the various thermodynamic quantities, have been in reference to an infinite system. It is of interest to see how they can be generalized to the case of finite systems. The linear size L of a finite body is an additional length scale and one would expect all thermodynamic quantities to depend only on the scaled variable $L/\xi(T)$. Note that

the correlation length $\xi(T) \sim \Delta T^{-\nu}$, where $\Delta T = T - T_c$, is the fundamental length scale in the critical region. Suppose that some physical quantity P varies as $(\Delta T)^{-\rho}$, in an infinite system, with exponent ρ (for example, α, β, etc.). The finite size scaling ansatz for P, in the limit of large L and small ΔT, is

$$P(T, L) \sim (\Delta T)^{-\rho} f_p\Big[\frac{L}{\xi(T)}\Big],$$

where f_p is some suitable function characteristic of the quantity P. The function $f_p(x)$ should approach unity as $x \rightarrow \infty$ so that $P(T, L)$ reduces to its bulk form as $L \rightarrow \infty$. When L is finite, all the physical quantities are analytic functions of the thermodynamic variables. Recall that the divergence or the anomalous behavior in the critical region is a manifestation of the infinite correlation length which exists only in an unbounded system. The analytic character of $P(T, L)$ is recovered if it is assumed that $f_p(x) \rightarrow x^{\rho/\nu}$ as $x \rightarrow 0$. Then, as $T \rightarrow T_c$, $P(T, L)$ varies as

$$P(T, L) \sim \Delta T^{-\rho}[L(\Delta T)^{\nu}]^{\rho/\nu} \sim L^{\rho/\nu}.$$

This result, which is a consequence of the finite scaling ansatz, shows that the behavior of P even in a finite system is determined by the critical exponents.

Taking P as the free energy density, one gets

$$F(T, L) \sim (\Delta T)^{2-\alpha} f_F\Big[L(\Delta T)^{\nu}\Big],$$

since $\rho = \alpha - 2$. If the field strength h is non-zero, the function f_F can be generalized as

$$F(T, L) \sim (\Delta T)^{2-\alpha} f_F\Big[\frac{h}{(\Delta T)^{\Delta}}, L(\Delta T)^{\nu}\Big].$$

This scaling form will be derived using renormalization group ideas in the following chapter.

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Chapter 4

Renormalization Group Theory

The scaling hypotheses for important thermodynamic quantities were introduced in the previous chapter. Then, scaling laws, which are relations among the critical exponents, were derived by determining the changes in physical quantities due to a change in length scale, and assuming that the correlation length is the only important length scale in critical region. Kadanoff's hypothesis about the equivalence of two models differing in short length scale features, but yielding identical long length scale properties, was also discussed there. Though this hypothesis could be used to motivate the scaling forms for the thermodynamic quantities, a number of adhoc assumptions were necessary. Nevertheless, it suggests that the averaging process, which smears out short length scale features, is something like a symmetry transformation for critical phenomena. Many characteristics of physical systems can be understood if the symmetries in the system are known. Effects of small perturbations which destroy the symmetries can also be classified using the characteristics of the unperturbed system. Spherically symmetric one particle quantum mechanical systems illustrate these points. The spherical symmetry of the potential yields the usual quantum numbers $(n, l, m, \cdots)$ for labeling the states, selection rules for transitions induced by a non-symmetric perturbing potential, etc. Thus, if Kadanoff transformations can be developed as symmetry operations for critical systems, it might become possible to extract many of their general features near the critical point. The renormalization group (RG) theory employs Kadanoff transformations and a change of spatial length scale to extract the properties near the critical point. Within this theoretical framework, the scaling hypotheses emerges in a very natural manner. Furthermore, it also provides methods for calculating the critical exponents. First of all, the RG ideas are elaborated below using the $1 - D$ Ising model.

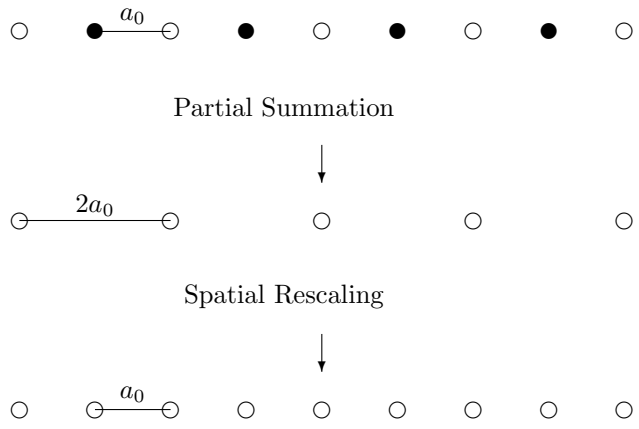

Figure 4.1: Steps involved in RG Transformation.

4.1 RG for 1-D Ising Model

The hamiltonian for the nearest neighbor 1-D Ising model is

$$\frac{H}{T} = -k\sum_j s_j s_{j+1} - h\sum_j s_j,$$

where $k = J/T$ and $h = h_{ex}/T$. The partition function for N spins is

$$Z_N = \sum_{config} \exp\Big(-\frac{H}{T}\Big).$$

The hamiltonian is completely specified by two parameters k and h. So, one can imagine a two dimensional parameter space, with each point in the space representing a certain hamiltonian. The free energy per spin is

$$F[H] = F(k,h) = -\frac{T}{N}\ln(Z_N).$$

A Kadanoff transformation can be performed by summing the configurations of every alternate spin variable in the calculation of Z_N. This procedure will give another Ising model (with parameters k' and h'), but with a lattice spacing which is twice the original spacing. Then, a change in the unit of length is introduced (in fact the length unit is doubled) so that the new model looks exactly like the old one. From the hamiltonian of the new model, it will be possible to determine the parameters k' and h' in terms of k and h. Thus the two operations, partial summation of spin configurations and spatial length rescaling, can be thought of as a transformation in the parameter space. The two operations are indicated in Figure 4.1.

The probability distribution of the spins is

$$P(\{s_i\}) = \frac{1}{Z}\exp(-\frac{H}{T}).$$

The probability distribution of the new model can be expressed as

$$P'(\{s_{2i+1}\}) = \sum_{con\ -\ i=even} P(\{s_i\}).$$

Note that the partition function computed from P' will be Z itself. Therefore the hamiltonian of the new model can be defined as

$$\exp\Big(-\frac{H'}{T}\Big) = \sum_{con\ -\ i=even} \exp\Big(-\frac{H}{T}\Big).$$

Defining $p(s_i, s_{i+1})$ as

$$p(s_i, s_{i+1}) = \exp\Big(ks_is_{i+1} + \frac{1}{2}hs_i\Big),$$

the Boltzmann factor $\exp(-H/T)$ becomes

$$\exp\Big(-\frac{H}{T}\Big) = \exp\Big(\frac{1}{2}hs_1\Big)\prod_i p(s_i, s_{i+1})\exp\Big(\frac{1}{2}hs_N\Big).$$

Now, let $p'(s_{i-1}, s_{i+1})$ be defined as

$$\begin{aligned} p'(s_{i-1}, s_{i+1}) &= \sum_{con\ -\ s_i} p(s_{i-1}, s_i)p(s_i, s_{i+1}) \\ &= 2\cosh[k(s_{i-1} + s_{i+1}) + h]\exp[\frac{h}{2}(s_{i-1} + s_{i+1})]. \end{aligned}$$

Then, assuming N to be odd, the new hamiltonian can be written as

$$\exp\Big(-\frac{H'}{T}\Big) = \exp\Big(\frac{h}{2}(s_1 + s_N)\Big)\prod_i p'(s_{2i-1}, s_{2i+1}).$$

Now, p' may be expressed as an exponential function so that the parameters k' and h' of H' can be readily identified. Note that when s_{2i-1} and s_{2i+1} take values ± 1, $(s_{2i-1} + s_{2i+1})$ takes values ± 2 and 0. Hence p' takes three distinct values. Therefore, three parameters are required to express p' as an exponential function. Thus p' may be expressed as

$$\begin{aligned} p'(s_{2i-1}, s_{2i+1}) &= 2\cosh[k(s_{2i-1} + s_{2i+1}) + h]\exp[\frac{h}{2}(s_{2i-1} + s_{2i+1})] \\ &= \exp[k's_{2i-1}s_{2i+1} + \frac{h'}{2}(s_{2i-1} + s_{2i+1}) + c'], \end{aligned} \tag{4.1}$$

where c' is the additional parameter. Then, H' can be written as an Ising model hamiltonian

$$\frac{H'}{T} = -k' \sum_j s_{2j-1} s_{2j+1} - h' \sum_j s_{2j-1} - (N-1)\frac{c'}{2}.$$

On renumbering the spin variables as $s_{2j-1} \rightarrow s_j$, H'/T takes the form

$$\frac{H'}{T} = -k' \sum_j s_j s_{j+1} - h' \sum_j s_j - (N-1)\frac{c'}{2}.$$

Then, Eq.(4.1), which defines the new parameters, is

$$\begin{aligned} &\exp\Big(k' s_i s_{i+1} + \frac{h'}{2}[s_i + s_{i+1}] + c'\Big) \\ &= \quad 2\cosh[k(s_i + s_{i+1}) + h] \exp[\frac{h}{2}(s_i + s_{i+1})]. \end{aligned} \tag{4.2}$$

Together with the constant term $(N-1)c'/2$, H' will yield the same partition function. The constant c' is simply the free energy (per spin) arising from the spins whose configurations have been summed up in obtaining H'. The definition of H' may be modified as

$$\exp\Big(-\frac{H'}{T} - N'c'\Big) = \sum_{con\ -\ i=even} \exp\Big(-\frac{H}{T}\Big),$$

where $N' \approx N/2$ since N is a large number. Then, H'/T is given by

$$\frac{H'}{T} = -k' \sum_j s_j s_{j+1} - h' \sum_j s_j,$$

and it is identical to the hamiltonian of the original Ising model. With this definition of H', the partition functions $Z'_{N'}$ and Z_N are related as

$$Z'_{N'} \exp(-N'c') = Z_N. \tag{4.3}$$

The equations for k' and h' can be obtained by substituting the possible values of s_i and s_{i+1} in Eq.(4.2). Putting $s_i = s_{i+1} = 1$, one gets

$$\exp(k' + h' + c') = 2\cosh(2k + h)\exp(h),$$

while the case $s_i = -1$ and $s_{i+1} = 1$ yields

$$\exp(-k' + c') = 2\cosh(h).$$

The same relation is obtained with $s_i = 1$ and $s_{i+1} = -1$. Finally, $s_i = s_{i+1} = -1$ leads to

$$\exp(k' - h' + c') = 2\cosh(2k - h)\exp(-h).$$

These equations can be easily manipulated to obtain expressions for k', h' and c'. The final results are

$$\begin{aligned}
\exp(4k') &= \cosh(2k+h)\frac{\cosh(2k-h)}{\cosh^2(h)}, \\
\exp(2h') &= \exp(2h)\frac{\cosh(2k+h)}{\cosh(2k-h)}, \\
\exp(4c') &= 2^4\cosh(2k+h)\cosh(2k-h)\cosh^2(h).
\end{aligned} \tag{4.4}$$

Thus, with any given values of k and h, k' and h' can be calculated. In other words, starting with an Ising model, a new Ising model can be derived by employing partial summation of the spin configurations. This procedure is usually known as *spin decimation*. However, the resulting lattice has a spacing $2a_0$. So the unit of length is doubled so that the numerical value of the new spacing is same as the original one. Due to this rescaling, any interval Δx in the original model becomes $\Delta x' = \Delta x/2$. Instead of decimating alternate spins, if every q spins are decimated and a spatial rescaling factor q is introduced, the relation obtained is $\Delta x = q\Delta x'$. In particular, the correlation length of the two systems are related as

$$\xi(k,h) = q\xi(k',h'). \tag{4.5}$$

Thus, spin decimation and spatial rescaling produce a new system with a smaller correlation length. Therefore, unless $\xi(k,h) = \infty$, this procedure drives the system away from the critical point. Since H and H' are identical in structure, the functional forms of $\xi(k,h)$ and $\xi(k',h')$ are the same.

Using the definition of free energy per spin,

$$F[H] = -\frac{T}{N}\ln(Z_N),$$

and Eq.(4.3), one gets

$$qF[H] - c' = F[H'], \tag{4.6}$$

where the general relation $qN' = N$ has been used. The factor q in this relation indicates that each site of the new model contains q sites of the old model. The parameter c' is the contribution (per site) from the decimated spins.

Flow Equations The equations connecting the parameter set (k', h') to (k, h) are known as flow equations. Symbolically, they are expressed as

$$\begin{aligned}
k' &= R_k(k,h), \\
h' &= R_h(k,h).
\end{aligned}$$

The equation for c' is not considered since it is explicitly given by k and h. Since $k = J/T$, these equations can also be rewritten in terms of T and h as

$$\begin{aligned} T' &= R_1(T,h), \\ h' &= R_2(T,h). \end{aligned}$$

Thus, the application of an RG transformation produces an identical system but at a different point in the parameter space. The fixed point (T^*, h^*) of the transformation is defined as

$$\begin{aligned} T^* &= R_1(T^*,h^*), \\ h^* &= R_2(T^*,h^*). \end{aligned}$$

By definition, the fixed point is unaltered by the transformation. That is, if $H = H^*$, then $H' = H^*$. However, the correlation length at the fixed point must satisfy Eq.(4.5),

$$\xi(T^*,h^*) = q\xi(T^*,h^*),$$

where q is the scale factor. This equation has solutions $\xi(T^*,h^*) = 0$ or ∞. The zero value is associated to a paramagnetic state while the infinite value corresponds to a system at the critical point. Thus (neglecting the solution $\xi(T^*,h^*) = 0$) the fixed point in the parameter space represents a critical system since the correlation length corresponding to that point is infinity. The flow equations can be linearised near the fixed point by writing

$$\begin{aligned} T' &= T^* + \Delta T', \\ h' &= h^* + \Delta h', \\ T &= T^* + \Delta T, \\ h &= h^* + \Delta h. \end{aligned}$$

In matrix notation, the linearised transformation equations are

$$\begin{pmatrix} \Delta T' \\ \Delta h' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta h \end{pmatrix} \tag{4.7}$$

where

$$\begin{aligned} R_{11} &= \frac{\partial R_1}{\partial T}^*, \quad R_{12} = \frac{\partial R_1}{\partial h}^*, \\ R_{21} &= \frac{\partial R_2}{\partial T}^*, \quad R_{22} = \frac{\partial R_2}{\partial h}^*. \end{aligned}$$

The deviations ΔT and Δh can be expressed as

$$\begin{pmatrix} \Delta T \\ \Delta h \end{pmatrix} = z_1 \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} + z_2 \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix} \tag{4.8}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the eigenvectors of the 2×2 matrix with eigenvalues $\rho_1(q)$ and $\rho_2(q)$. The combining coefficients z_1 and z_2 can be expressed in terms of ΔT and Δh. Then the linearised RG equations can also be written as

$$\begin{pmatrix} \Delta T' \\ \Delta h' \end{pmatrix} = z_1\rho_1(q) \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} + z_2\rho_2(q) \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix}. \tag{4.9}$$

If the transformation is applied once again, one gets $\Delta T''$ and $\Delta h''$ from $\Delta T'$ and $\Delta h'$. That is,

$$\begin{pmatrix} \Delta T'' \\ \Delta h'' \end{pmatrix} = z_1\rho_1^2(q) \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} + z_2\rho_2^2(q) \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix}$$

However, $\Delta T''$ and $\Delta h''$ could have been obtained from ΔT and Δh by decimating q^2 spins and then employing a spatial rescaling by q^2. Thus one can also write

$$\begin{pmatrix} \Delta T'' \\ \Delta h'' \end{pmatrix} = z_1\rho_1(q^2) \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} + z_2\rho_2(q^2) \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix}.$$

The two expressions obtained for $\Delta T''$ and $\Delta h''$ show that

$$\rho_i(q^2) = \rho_i^2(q), i = 1, 2.$$

For this relation to be obeyed for arbitrary q, one should have

$$\rho_i(q) = q^{y_i}, i = 1, 2.$$

Thus the dependence of the eigenvalues on the parameter q is obtained.

To work out the eigenvalues corresponding to the flow equations (4.4), it is more appropriate to define a reduced temperature

$$T_0 = exp(-4k) = exp(-\frac{4J}{T}).$$

Thus at $T = 0,\ T_0 = 0$ and at $T = \infty,\ T_0 = 1$. The reduced temperature T_0 and h are then the parameters characterizing the hamiltonian. In terms of T_0 and h, the flow equations reduce to

$$\begin{aligned} T_0' &= 4T_0 \frac{\cosh^2(h)}{1 + T_0 + 2T_0\cosh(h)}, \\ h' &= h + \frac{1}{2}\ln\Big(\frac{e^h + e^{-h}T_0}{e^{-h} + e^h T_0}\Big). \end{aligned}$$

The fixed point values are $h^* = 0$, and $T_0^* = 0$ and $T_0^* = 1$. The last value corresponds to $T = \infty$ and hence represents the paramagnetic state. Thus $(T_0^*, h^*) = (1, 0)$ is a trivial fixed point which is of no interest. The point $(T_0^*, h^*) = (0, 0)$ is the non-trivial fixed point. The elements of the matrix

in Eq.(4.7) are given by $R_{11} = 4, R_{12} = 0, R_{21} = 0$ and $R_{22} = 2$. Thus the transformation matrix is diagonal and hence its eigenvalues are

$$\begin{aligned} \rho_1(q=2) &= 4 = 2^2, \\ \rho_2(q=2) &= 2 = 2^1, \end{aligned}$$

which yield $y_1 = 2$ and $y_2 = 1$. With the usual set of eigenvectors, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, Eq.(4.8) yields $z_1 = \Delta T_0 = T_0$ and $z_2 = \Delta h = h$. Thus, in the neighborhood of the fixed point $(T_0^*, h^*) = (0, 0)$, the RG transformation (4.9) can be written as

$$\begin{aligned} T_0' &= 2^2 T_0 \\ h' &= 2^1 h. \end{aligned}$$

Generalizing to the case of arbitrary scale factor q, one gets

$$\begin{aligned} T_0' &= q^2 T_0 \\ h' &= q^1 h. \end{aligned}$$

Note that these are same as the heuristic relations proposed in Kadanoff's derivation of scaling behaviour. In the notation used there, $x = y_2 = 1$ and $y = y_1 = 2$.

The relation between the correlation lengths of the two systems is

$$\xi(T_0, h) = q\xi(T_0', h') = q\xi(q^{y_1} T_0, q^{y_2} h).$$

Now, consider the case $h = 0$, that is, the original system is without any field. Then ξ is given by

$$\xi(T_0) = q\xi(q^{y_1} T_0).$$

The solution of this functional equation can be obtained by giving a special value to q which is arbitrary. Putting $q = T_0^{-1/y_1}$, it is found that

$$\xi(T_0) = T_0^{-1/y_1} \xi(1) = T_0^{-\nu} \xi(1).$$

Thus, the parameter y_1 can be identified as ν^{-1} where ν is the correlation length exponent. This relation also shows that

$$\xi(T) = \xi(T_0) \propto \exp(+4J\nu/T).$$

Thus, for the 1-D Ising model, an exponential divergence is obtained, as compared to the usual power law divergence.

Eq.(4.6) for free energy (per spin) becomes

$$F(T_0, h) = q^{-1} F(T_0', h') = q^{-1} F(q^{y_1} T_0, q^{y_2} h),$$

where the term c' is omitted. The effect of that term will be discussed later. Once again, the choice $q = T_0^{-1/y_1}$ yields

$$F(T_0, h) = (T_0)^{1/y_1} F\Big(1, \frac{h}{T_0^{y_2/y_1}}\Big),$$

which is of the scaling form if

$$\frac{1}{y_1} = 2 - \alpha, \ \frac{y_2}{y_1} = \Delta.$$

The detailed analysis of this model has, thus, shown that divergence of physical quantities and scaling behaviour can be understood in terms of the RG approach. The RG transformation sets the system in motion in the parameter space and so this approach recasts the problem of critical behaviour as a dynamical problem.

4.2 General Renormalization Group

A specific model of a system is defined by a hamiltonian which contains several parameters like a_2, a_4, c and h in the Landau-Ginzburg hamiltonian. Thus a parameter space is imagined and the state of the system is represented as a point in this space. The RG approach is based on the observation that two models of critical behaviour are equivalent if they differ only in the short length scale variations of microscopic variables (like spin variables). So one starts by investigating how the parameters in a coarse grained model are related to those in a detailed model. If the coarse grained model has the same structure of the detailed model, then it is also represented by a point in the parameter space. As explained in the example of 1-D Ising model, the RG transformation takes a point in the parameter space to another point. This picture then naturally leads to the concept of a fixed point in parameter space, the hamiltonian corresponding to the fixed point being invariant under the RG transformation. With a slight generalization of the concepts used in the example of 1-D Ising model, the scaling theory and universality of critical phenomena can be understood in terms of the properties of the RG transformation near a fixed point. The three steps involved in a general RG transformation are discussed below.

4.2.1 Reduction of Degrees of Freedom

In general, a coarse grained model is obtained by averaging the microscopic variables over a certain local region. Therefore, coarse graining leads to a reduction in the number of degrees of freedom. Recall that the cell averaging procedure, introduced by Kadanoff, led to the scaling hypotheses. Now,

consider a rigorous method to obtain the cell hamiltonian. This method was explained earlier, but is repeated here with slight generalization. The cell spin variable is defined as the average of the spins belonging to a cell, that is

$$s^{\mu}_{\mathbf{x}}{}'' = \frac{1}{q^d} < \sum_{\mathbf{x}\in c} s^{\mu}_{c} >,$$

where the symbol $\sum_{\mathbf{x}\in c}$ indicates that the summation is over all lattice points $\mathbf{x}$ belonging to a cell of size qa_0, where a_0 is the lattice spacing. Note that each cell contains q^d spins, and the index μ denotes the spin component. For describing the spin variations over distances larger than qa_0, these cell averaged variables are expected to be adequate. However, they can not describe spin variations over distances smaller than qa_0. This feature is similar to that in an Ising model where the cut-off length for spin variations is the lattice spacing a_0. Similarly, the spatially discrete Landau-Ginzburg model hamiltonian

$$\frac{H}{T} = b^d \sum_{\mathbf{x}} \Big[a_0 + a_2 \mathbf{s}^2_{\mathbf{x}} + a_4 \mathbf{s}^4_{\mathbf{x}} - h s^1_{\mathbf{x}} - \frac{c}{b^2} \sum_{\mathbf{y}} (\mathbf{s}_{\mathbf{x}} - \mathbf{s}_{\mathbf{y}})^2 \Big],$$

or its continuous version

$$\frac{H[s]}{T} = \int_V \Big(a_0 + a_2 \mathbf{s}^2(\mathbf{x}) + a_4 \mathbf{s}^4(\mathbf{x}) - h s_1(\mathbf{x}) + c[\nabla \mathbf{s}(\mathbf{x})]^2 \Big) d\mathbf{x},$$

accounts for spin variations over length scales larger than a cut-off value $b = 2\pi/\Lambda$.

Since the cell spin is defined as a sum over the spins in a cell, the probability distribution of the cell variables is

$$\begin{aligned} P''[s''] &= < \prod_{\mu\ \mathbf{x}} \delta\Big(s^{\mu}_{\mathbf{x}}{}'' - q^{-d} \sum_{\mathbf{x}\in c} s^{\mu}_{\mathbf{x}} \Big) > \\ &= \frac{1}{Z} \int \exp\Big(-\frac{H[s]}{T} \Big) \prod_{\mu\ \mathbf{x}} \delta\Big(s^{\mu}_{\mathbf{x}}{}'' - q^{-d} \sum_{\mathbf{x}\in c} s^{\mu}_{\mathbf{x}} \Big) \prod_{\mu\ \mathbf{x}} ds^{\mu}_{\mathbf{x}}, \end{aligned}$$

where $H[s]$ and Z are respectively the hamiltonian and partition function of the Ising model or the L-G model. In writing this equation, the definition of the distribution of sum of random variables has been used. The hamiltonian $H''[s'']$ and partition function Z'' of the cell model are defined as

$$P''[s''] = \frac{1}{Z''} \exp\Big(-\frac{H''[s'']}{T} \Big).$$

Note that $Z'' = Z$. Equivalently, $H''[s'']$ can be defined as

$$\exp\Big(-\frac{H''[s'']}{T} \Big) = \int \exp\Big(-\frac{H[s]}{T} \Big) \prod_{\mu\ \mathbf{x}} \delta\Big(s^{\mu}_{\mathbf{x}}{}'' - q^{-d} \sum_{\mathbf{x}\in c} s^{\mu}_{\mathbf{x}} \Big) \prod_{\mu\ \mathbf{x}} ds^{\mu}_{\mathbf{x}}.$$

The two probability distributions $P[s]$ and $P''[s'']$ or the hamiltonians $H[s]$ and $H''[s'']$ are equivalent as far as spin variations over distances greater than qa_0 are concerned. Obtaining $H''[s'']$ from a given $H[s]$ is called a Kadanoff transformation K_q. In a symbolic manner, it may be written as

$$H''[s''] = K_q H[s],$$

where q is the cell size parameter. With $q = 1$, one gets the identity transformation K_1. If another Kadanoff transformation $K_{q'}$ is performed, one gets

$$K_{q'} H''[s''] = K_{q'} K_q H[s].$$

Two transformations lead to a cell size parameter $q'q$. Therefore one can also write

$$K_{q'} H''[s''] = K_{q'\ q} H[s].$$

Thus the transformations $\{K_q\}$ have the property

$$K_{q'} K_q = K_{q'q}.$$

This defines the multiplication law for the operators $\{K_q\}$. The operator K_q does not have an inverse since H can not be obtained from H''. Therefore, the operators $\{K_q\}$ are said to form a semigroup. Note that the Kadanoff transformation can not produce any singular behaviour in the thermodynamic quantities since the long wave length variations of spins are unaltered by it.

For the spatially continuous L-G model, $H[s]$ can be expressed in terms of the Fourier amplitudes $s_{i\mathbf{k}}$ of spin component $s_i(\mathbf{x})$. Now, $s_i(\mathbf{x})$ contains Fourier modes with k in 0 to $\Lambda = 2\pi/b$ or wave length from b to ∞. The aim is to introduce a coarse graining so that the cut-off wavelength is qb or the cut-off value of k is Λ/q. Thus, $P''[s'']$ is obtained by integrating out the Fourier amplitudes with k in Λ/q to Λ in $P[s]$. That is, the hamiltonian $H''[s'']$ is given by

$$\exp\Big(-\frac{H''[s'']}{T}\Big) = \int \exp\Big(-\frac{H[s]}{T}\Big) \prod_{i\ \ \Lambda/q<k\leq\Lambda} ds_{i\mathbf{k}}.$$

Then, $H''[s'']$ (and hence $P''[s'']$) will contain Fourier amplitudes with $k \leq \Lambda/q$. Thus it describes spin variations with a wave length greater than qb. That is, $s_i''(\mathbf{x})$ can be expressed as

$$s_i''(\mathbf{x}) = \frac{1}{L^{d/2}} \sum_{k\leq\Lambda/q} \exp(\imath\mathbf{k}\cdot\mathbf{x}) s_{i\mathbf{k}}.$$

The Kadanoff transformation so defined can be symbolically written as

$$H''[s_{i\mathbf{k}}, k \leq \Lambda/q] = K_q H[s_{i\mathbf{k}}, k \leq \Lambda].$$

4.2.2 Spatial Rescaling

The L-G hamiltonian is characterized in terms of the parameter set $\mu = (a_0, a_2, a_4, c)$. So a particular point in the μ-space (parameter space) represents a certain hamiltonian H. The Kadanoff transformation K_q yields a new hamiltonian H''. It is hoped that H'' can also be written in the L-G form and hence can be represented by a new point $\mu'' = (a_0'', a_2'', a_4'', c'')$ in the μ-space. There is a basic length scale $b = 2\pi/\Lambda$ associated with H. The length scale associated with H'' is qb. The parameter set $\mu = (a_0, a_2, a_4, c)$ of H represents the details of the system over the basic length scale b just as the parameter $k = J/T$ of the Ising model is characteristic of the details of the system over the lattice unit a_0. To compare the two parameter sets μ and μ'', it is necessary to make both of them represent the details of the system over the same basic length scale. So for the coarse grained model H'', the length unit is taken to be q times the unit of length in the original model H. That is, the coarse grained system should be viewed with a coarser length unit. Due to this change of length unit, physical quantities acquire new numerical values. For example, the spatial coordinate x is changed to $x' = x/q$. Similarly, the size of the system L changes to $L' = L/q$. The spin variable $s_i''(\mathbf{x})$ in H'' gets altered to $s_i(\mathbf{x}')$. The volume integral $\int_V d\mathbf{x}$ is transformed to $\int_{V'} d\mathbf{x}' = q^{-d} \int_V d\mathbf{x}$. The wave vector $\mathbf{k}$ becomes $\mathbf{k}' = q\mathbf{k}$. Thus the spin variable $s_i''(\mathbf{x})$, in H'', should be replaced by $s_i(\mathbf{x}')$ before making a comparison of the two parameter sets. The Fourier component $s_{i\mathbf{k}}$ was defined as

$$s_{i\mathbf{k}} = \frac{1}{L^{d/2}} \int_V \exp(-\imath \mathbf{k} \cdot \mathbf{x}) s_i''(\mathbf{x}) d\mathbf{x}.$$

With the coarse unit, one finds

$$s_{i\mathbf{k}} \to \frac{q^d}{L^{d/2} q^{d/2}} \int_{V'} \exp(-\imath q \mathbf{k} \cdot \mathbf{x}') s_i(\mathbf{x}') d\mathbf{x}' \;=\; q^{d/2} s_{i\ q\mathbf{k}}.$$

That is, all Fourier components $s_{i\mathbf{k}}$ ($k \leq \Lambda/q$) in H'' should be replaced by $q^{d/2} s_{i\mathbf{k}'}$ where $\mathbf{k}' = q\mathbf{k}$. Together with this change, all functions which depend on $\mathbf{k}$ should be expressed in terms of $\mathbf{k}'$ and factors like L^d should be expressed in terms of $L' = L/q$.

4.2.3 Rescaling of Spin Variables

The aim of employing the coarse graining operation is to exploit the RG transformation as a symmetry transformation for critical phenomena. Therefore, it becomes necessary to locate a fixed point of the transformation. As shown below, it is necessary to rescale the magnitude of the spin variables in the

coarse grained model so that a proper fixed point can be isolated. That is, it becomes necessary to replace $s_i(\mathbf{x}')$ or $s_{i\mathbf{k}'}$ with $\alpha(q)s(\mathbf{x}')$ or $\alpha(q)s_{i\mathbf{k}'}$. Thus, after the reduction of degrees of freedom, the replacement to be made is

$$s_i''(\mathbf{x}) \rightarrow \alpha(q)s_i(\mathbf{x}'),\ \mathbf{x}' = \mathbf{x}/q.$$

In terms of Fourier amplitudes, this means

$$s_{i\mathbf{k}} \rightarrow \alpha(q)q^{d/2}s_{i\mathbf{k}'}\ ,\ \mathbf{k}' = q\mathbf{k}.$$

The parameter $\alpha(q)$ is to be determined such that a proper fixed point of the transformation can be identified. For the example of $1-D$ Ising model, $\alpha(q)$ was unity.

The three steps described can be symbolically represented as

$$\begin{aligned} H''[s''] &= K_q H[s], \\ H'[s] &= H''[s''] \\ \mathbf{s}''(\mathbf{x}) &\rightarrow \alpha(q)\mathbf{s}(\mathbf{x}'),\ \mathbf{x}' = \mathbf{x}/q. \end{aligned}$$

In the discrete model, the combined operation is

$$\exp\Big(-\frac{H'[s]}{T}\Big) = \int \exp\Big(-\frac{H[s]}{T}\Big) \prod_{\mu\ \mathbf{x}'} \delta\Big(\alpha(q)s^{\mu}_{\mathbf{x}'} - q^{-d}\sum_{\mathbf{x}\in c} s^{\mu}_{\mathbf{x}}\Big) \prod_{\mu\ \mathbf{x}} ds^{\mu}_{\mathbf{x}}.$$

In the continuum model, using Fourier amplitudes, the combined operation becomes

$$\exp\Big(-\frac{H'[\{s_{i\mathbf{k}'}\}]}{T}\Big) = \Big[\int \exp\Big(-\frac{H[\{s_{i\mathbf{k}}\}]}{T}\Big) \prod_{i\ \Lambda/q<k\leq\Lambda} ds_{i\mathbf{k}}\Big],$$

together with the replacement

$$s_{i\mathbf{k}} \rightarrow \alpha(q)q^{d/2}s_{i\mathbf{k}'}\ ,\mathbf{k}' = q\mathbf{k}.$$

In the μ-space, RG transformation takes the point μ to μ' and is represented as

$$\mu' = \mathbf{R}_q\mu,$$

where $\mathbf{R}_q$ denotes the three steps described above.

Just as K_q, $\mathbf{R}_q$ also should satisfy the multiplication law $\mathbf{R}_{q'}\mathbf{R}_q = \mathbf{R}_{q'q}$. This is required since one can reach the point μ'' from μ in two equivalent ways, $\mu'' = \mathbf{R}_{q'}\mu' = \mathbf{R}_{q'}\mathbf{R}_q\mu$ and $\mu'' = \mathbf{R}_{q'q}\mu$. The first involves repeated coarse graining over cell sizes qb and $q'b$ while the second consists of a single coarse graining operation over the cell size $q'qb$. Since the results of the two ways must be the same, the multiplication law for $\mathbf{R}_q$ is a necessary condition. In terms of Fourier amplitudes, one notes that $\mathbf{R}_{q'}\mathbf{R}_q$ involves repeated elimination of

modes with k in Λ/q to Λ and then in $\Lambda/q'q$ to Λ/q which amounts to applying a single transformation $\mathbf{R}_{q'q}$. This requirement puts a restriction on the scale parameter $\alpha(q)$,

$$\alpha(q')\alpha(q) = \alpha(q'q).$$

Therefore, the parameter $\alpha(q)$ should be of the form

$$\alpha(q) = q^a,$$

where a is some number to be adjusted to isolate a fixed point of the RG transformation $\mathbf{R}_q$. For the sake of clarifying the ideas, the three steps are applied to the Gaussian model in the following section.

4.3 Gaussian Model - RG Steps

The hamiltonian of the Gaussian model is

$$\frac{H[s]}{T} = \int_V \{a_2 \mathbf{s}^2(\mathbf{x}) - h s_1(\mathbf{x}) + c[\nabla \mathbf{s}(\mathbf{x})]^2\} d\mathbf{x}.$$

The parameter space is $\mu = (a_2, c, h)$. In terms of Fourier components, H/T is

$$\frac{H[s]}{T} = \sum_{ik \le \Lambda} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 - L^{d/2} h s_{10}.$$

The first step of RG yields

$$\begin{aligned}
&\exp\Big(-\frac{H''[s'']}{T}\Big) \\
&= \int \exp\Big[-\sum_{i\ k\le\Lambda} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 + L^{d/2} h s_{10}\Big] \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}} \\
&= \exp\Big[-\sum_{i\ k\le\Lambda/q} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 + L^{d/2} h s_{10}\Big] \\
&\times \int \exp\Big[-\sum_{i\ \Lambda/q<k\le\Lambda} (a_2 + ck^2)|s_{i\mathbf{k}}|\Big] \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}},
\end{aligned}$$

The last integral, (I) can be evaluated as follows. Let the region defined as $\Lambda/q < k \le \Lambda$ be divided into two symmetric parts and Δ denote one of them. Then

$$\begin{aligned}
I &= \int \exp\Big[-\sum_{i\ k\in 2\Delta} (a_2 + ck^2)|s_{i\mathbf{k}}|^2\Big] \prod_{i\ k\in 2\Delta} ds_{i\mathbf{k}} \\
&= 2\int \exp\Big[-2\sum_{i\ k\in\Delta} (a_2 + ck^2)(s^{R\ 2}_{i\mathbf{k}} + s^{I\ 2}_{i\mathbf{k}})\Big] \prod_{i\ k\in\Delta} ds^R_{i\mathbf{k}} ds^I_{i\mathbf{k}} \\
&= \prod_{i\ k\in\Delta} \Big[\frac{\pi}{a_2 + ck^2}\Big]^{1/2} \Big[\frac{\pi}{a_2 + ck^2}\Big]^{1/2} = \Big[\prod_{k\le\Lambda/q} \frac{\pi}{a_2 + ck^2}\Big]^{n/2}.
\end{aligned}$$

Therefore H'' is given by

$$\frac{1}{T}H''[\{s_{i\mathbf{k}}\}, k \leq \Lambda/q] = \sum_{i \ k\leq\Lambda/q} (a_2 + ck^2)|s_{i\mathbf{k}}|^2 - L^{d/2}hs_{10} + A,$$

where A is a constant and

$$A = -\frac{n}{2}\sum_{\Lambda/q<k\leq\Lambda} \ln\Big(\frac{\pi}{a_2 + ck^2}\Big).$$

Thus, the elimination of modes with k in Λ/q to Λ yields the hamiltonian H'' in the Gaussian form, but with an additional constant term A. This term is the contribution to the total free energy from the eliminated modes.

The second step is to replace $s_{i\mathbf{k}}$ with $q^{a+d/2}s_{i\mathbf{k}'}$ where $\mathbf{k}' = q\mathbf{k}$. Thus, H' is obtained from H''. The parameters k^2 and L also should be expressed in terms of k' and L', i.e. $k^2 = q^{-2}k'^{\ 2}$ and $L = qL'$. Thus one obtains

$$\frac{H'}{T} = \sum_{i \ k'\leq\Lambda} (a_2 + cq^{-2}k'^{\ 2})q^{2a+d}|s_{i\mathbf{k}'}|^2 - (qL')^{d/2}hq^{a+d/2}s_{10} + A,$$

which can be written as

$$\frac{H'}{T} = \sum_{i \ k'\leq\Lambda} (a_2' + c'k'^{\ 2})|s_{i\mathbf{k}'}|^2 - L'^{\ d/2}h's_{10} + A,$$

where the new parameter set $\mu' = (a_2', c', h')$ is given by

$$\begin{aligned} a_2' &= a_2 q^{2a+d}, \\ c' &= cq^{2a+d-2}, \\ h' &= hq^{a+d}. \end{aligned}$$

These transformations are symbolically represented as $\mu' = \mathbf{R}_q\mu$. For the Gaussian model, $\mathbf{R}_q$ is a diagonal matrix. The parameter $\alpha(q) = q^a$ is to be adjusted to find proper fixed points (defined as $\mu^* = \mathbf{R}_q\mu^*$) of the transformation. The choice $a = 0$ leads to the fixed point values $a_2^* = c^* = h^* = 0$ (for $d \neq 2$). Another choice is $a = (2-d)/2$ which yields $a_2^* = h^* = 0$ and c^* is arbitrary. Thus the fixed point hamiltonian is

$$\frac{H^*}{T} = c^* \int (\nabla\mathbf{s})^2 d\mathbf{x}.$$

Yet another choice is $a = -d/2$ so that $c^* = h^* = 0,\ a_2^*$ is arbitrary and hence

$$\frac{H^*}{T} = a_2^* \int \mathbf{s}^2 d\mathbf{x}.$$

This fixed point hamiltonian will not show any spatial correlation since the gradient term is absent. Later, it will be shown that the second choice yields the exponents of Gaussian model.

4.4 Few Points about RG

Some of the important points regarding the RG transformation for the L-G model are discussed below.

(i) The averages calculated with the distributions

$$P = \frac{1}{Z}\exp(-\frac{H}{T}),$$

and

$$P' = \frac{1}{Z}\exp(-\frac{H'}{T}),$$

are equivalent. That is,

$$< s_i(\mathbf{x}) >_P = \alpha(q) < s_i(\mathbf{x}') >_{P'} = q^a < s_i(\mathbf{x}') >_{P'} \quad , \mathbf{x}' = \mathbf{x}/q.$$

Similarly the spatial correlation functions computed with P and P' are related as

$$< s_i(\mathbf{x})\ s_j(\mathbf{x}+\mathbf{r}) >_P = q^{2a} < s_i(\mathbf{x}')\ s_j(\mathbf{x}'+\mathbf{r}') >_{P'} \quad , \mathbf{r}' = \mathbf{r}/q.$$

The Fourier transform of the correlation function (for $n = 1$) is

$$G(k,\mu) = < |s_{\mathbf{k}}|^2 >_P .$$

Here and in what follows, the parameter μ in the argument denotes the dependence of G (and other averages) on H. The RG transformation yields $\mu' = \mathbf{R}_q\mu$ and $s_{\mathbf{k}}$ is to be replaced by $q^{a+d/2}s_{\mathbf{k}'}$ where $\mathbf{k}' = q\mathbf{k}$. Therefore one gets,

$$G(k,\mu) = q^{2a+d}G(qk,\mu').$$

(ii) The transformation for the field term can be obtained easily. Note that the L-G hamiltonian can be written as

$$\frac{H[s]}{T} = \frac{H_0[s]}{T} - L^{d/2}hs_{10},$$

where H_0 is an even functional of $\mathbf{s}(\mathbf{x})$, i.e. changing $\mathbf{s}(\mathbf{x})$ to $-\mathbf{s}(\mathbf{x})$ leaves H_0 unchanged. In terms of Fourier components $s_{i\mathbf{k}}$, this means that if every $s_{i\mathbf{k}}$ is replaced by $-s_{i\mathbf{k}}$, then $H_0[\{s_{i\mathbf{k}}\}]$ is unchanged. The first step of RG defines the new hamiltonian H'' as

$$\begin{aligned}\exp\Big(-\frac{H''[s'']}{T}\Big) &= \int \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}} \exp\Big(-\frac{H_0[s]}{T} + L^{d/2}hs_{10}\Big) \\ &= \exp(L^{d/2}hs_{10})\exp\Big(-\frac{H_0''[\{s_{i\mathbf{k}}\}]}{T}\Big),\ k \le \Lambda/q.\end{aligned}$$

Note that H_0'' is unaltered by changing $s_{i\mathbf{k}}$ to $-s_{i\mathbf{k}}$ ($k \leq \Lambda/q$). The second and third steps (yielding H' from H''), then, amounts to replacing $s_{i\mathbf{k}}$ with $q^{a+d/2}s_{i\mathbf{k}'}$ where $\mathbf{k}' = q\mathbf{k}$. Thus H' becomes

$$\frac{1}{T}H'[\{s_{i\mathbf{k}'}\},\ k' \leq \Lambda] = \frac{1}{T}H_0'[\{s_{i\mathbf{k}'}\}, k' \leq \Lambda] - L'^{\,d/2}hq^{a+d}s_{10}.$$

Thus the magnetic field term is altered as

$$h' = hq^{a+d}.$$

Therefore, it is only necessary to work out the form of H_0'' for implementing RG for the L-G model.

(iii) In discussing the RG transformation, it was assumed that H'/T can be written i. the same form as H/T. But for the appearance of a constant term (which is the contribution to the free energy from the eliminated modes), this was found to be possible for the 1-D Ising model and the Gaussian model. However, it will be shown later that if one starts with the L-G hamiltonian, the same form is not retained when degrees of freedom are reduced. In fact, it is found that many additional terms containing $\mathbf{s}^6, \mathbf{s}^8, (\nabla^2\mathbf{s})^2$, etc. are generated in the process. This feature appears to create a serious difficulty in using the RG ideas for the L-G model. A way to circumvent this problem is to start with a rather general hamiltonian. For example, one may consider

$$\begin{aligned}\frac{H[s]}{T} &= \int_V \Big[a_0 + a_2\mathbf{s}^2(\mathbf{x}) + a_4\mathbf{s}^4(\mathbf{x}) + a_6\mathbf{s}^6(\mathbf{x}) + \cdots - hs_1(\mathbf{x}) \\ &+ [\nabla\mathbf{s}(\mathbf{x})]^2\{c_0 + c_2\mathbf{s}^2(\mathbf{x}) + c_4\mathbf{s}^4(\mathbf{x}) + c_6\mathbf{s}^6(\mathbf{x}) + \cdots\} \\ &+ [\nabla^2\mathbf{s}(\mathbf{x})]^2\{d_0 + d_2\mathbf{s}^2(\mathbf{x}) + d_4\mathbf{s}^4(\mathbf{x}) + d_6\mathbf{s}^6(\mathbf{x}) + \cdots\} + \cdots\Big]d\mathbf{x}.\end{aligned}$$

where, but for the magnetic field term, the hamiltonian is an even functional of $\mathbf{s}(\mathbf{x})$. Then the parameter space

$$\mu = (h, \{a_j\}, \{c_j\}, \{d_j\}, \cdots),\ j = 0, 1, \cdots,$$

is infinite dimensional and this is one of the difficulties in implementing the RG procedure. But as shown later, $\mu = (h, a_0, a_2)$ is adequate for $d \geq 4$, while for $d < 4$ it is necessary to consider $\mu = (h, a_0, a_2, a_4)$.

(iv) For spatially discrete models, the parameter q has to be an integer. Thus, one may take $q = 2$ and then, using the property

$$\mathbf{R}_{q'}\mathbf{R}_q = \mathbf{R}_{q'q},$$

obtain the general $\mathbf{R}_q$ with $q = 2^l$ as

$$\mathbf{R}_q = \mathbf{R}_{2^l} = [\mathbf{R}_2]^l.$$

For integrating out the Fourier components, q can be any positive number.

4.5 Fixed Point and Critical Surface

The RG transformation, symbolically written as $\mu' = \mathbf{R}_q \mu$, means the transformation of the parameters of the hamiltonian

$$\begin{aligned} h' &= hq^{a+d}. \\ a'_\alpha &= u_\alpha(\{a_\beta\}, \{c_\beta\}, \{d_\beta\}, \cdots), \\ c'_\alpha &= v_\alpha(\{a_\beta\}, \{c_\beta\}, \{d_\beta\}, \cdots), \\ d'_\alpha &= w_\alpha(\{a_\beta\}, \{c_\beta\}, \{d_\beta\}, \cdots), etc. \end{aligned}$$

The fixed point of the transformation μ^* is defined as

$$\mu^* = \mathbf{R}_q \mu^*,$$

for all q . Thus μ^* is an infinite dimensional vector given by

$$\mu^* = (0, \{a^*_\beta\}, \{c^*_\beta\}, \{d^*_\beta\}, \cdots),$$

where the entry 0 is the fixed point value of the field. For the 1-D Ising model and the Gaussian model, the fixed points have been obtained. The existence of a fixed point for any particular (nontrivial) model has not been proved. However, certain fixed points can be isolated by an approximate implementation of RG transformation.

A critical surface in the parameter space is defined as the surface (or set of points) formed by all μ's such that

$$\lim_{q\to\infty} \mathbf{R}_q \mu = \mu^*.$$

In the neighborhood of a fixed point, one may write

$$\begin{aligned} \mu &= \mu^* + \delta\mu, \\ \mu' &= \mu^* + \delta\mu', \end{aligned}$$

and define a linearised RG transformation

$$\delta\mu' = \widetilde{\mathbf{R}}_q \delta\mu.$$

If $\mathbf{R}_q$ and μ^* are known, $\widetilde{\mathbf{R}}_q$ can be obtained. In fact, the explicit form of the linearised transformation is

$$\begin{aligned} h' &= hq^{a+d}. \\ \delta a'_\alpha &= \sum_\beta \Big(\frac{\partial u_\alpha}{\partial a_\beta}\Big)_* \delta a_\beta + \Big(\frac{\partial u_\alpha}{\partial c_\beta}\Big)_* \delta c_\beta, \cdots, etc. \end{aligned}$$

Therefore, $\widetilde{\mathbf{R}}_q$ has a block structure

$$\widetilde{\mathbf{R}}_q = \begin{pmatrix} q^{a+d} & \vec{0} \\ \vec{0} & \widetilde{\mathbf{R}}_{0q} \end{pmatrix},$$

where $\widetilde{\mathbf{R}}_{0q}$ represents the block corresponding to the even part of the hamiltonian. The eigenvalues (ρ_j) and the eigenvectors ($\mathbf{e}_j$) of $\widetilde{\mathbf{R}}_q$ are defined as

$$\widetilde{\mathbf{R}}_q \mathbf{e}_j = \rho_j \mathbf{e}_j.$$

The eigenvalues depend on the parameter q. If the μ space is infinite dimensional, j is infinite in number. Due to the block structure, one of the eigenvalues is $\rho_h = q^{a+d}$. A linear space can be formed in the vicinity of the fixed point μ^*. Assuming that the eigenvectors form a basis for this linear space, any arbitrary deviation $\delta\mu = \mu - \mu^*$ can be expressed as

$$\delta\mu = h\mathbf{e}_h + \sum_j t_j \mathbf{e}_j.$$

Then $\delta\mu'$ becomes

$$\begin{aligned} \delta\mu' &= \mu' - \mu^* = \widetilde{\mathbf{R}}_q \delta\mu, \\ &= h\rho_h(q)\mathbf{e}_h + \sum_j t_j \rho_j(q)\mathbf{e}_j. \end{aligned}$$

A second application of $\widetilde{\mathbf{R}}_q$ yields

$$\delta\mu'' = \widetilde{\mathbf{R}}_q \delta\mu' = \widetilde{\mathbf{R}}_q \widetilde{\mathbf{R}}_q \delta\mu = \widetilde{\mathbf{R}}_{q^2} \delta\mu.$$

In obtaining the last equality, the multiplication property of $\mathbf{R}_q$ and hence that of $\widetilde{\mathbf{R}}_q$, has been used. In terms of eigenvalues, this property implies

$$[\rho_j(q)]^2 = \rho_j(q^2) \ , j = 1, 2, \cdots.$$

Therefore, the q dependence of the eigenvalues should be of the form

$$\rho_j(q) = q^{y_j} \ , j = 1, 2, \cdots,$$

where y_j is independent of q. Note that the eigenvalue ρ_h has this power law dependence on q. Now, assume that $\rho_j < 1$, for $j > j_0$, i.e. $y_j < 0$ for $j > j_0$. Further, let μ_c be a point in the vicinity of μ^* so that

$$\mu_c = \mu^* + \delta\mu = \mu^* + \sum_{j>j_0} t_j \mathbf{e}_j.$$

On repeatedly applying $\widetilde{\mathbf{R}}_q$ l times, one gets

$$[\widetilde{\mathbf{R}}_q]^l \mu_c = \mu^* + \sum_{j>j_0} t_j q^{l y_j} \mathbf{e}_j.$$

Therefore, $\mu_c \to \mu^*$ as $l \to \infty$. In other words, any μ which lies in the sub-space spanned by $\mathbf{e}_j$, $j > j_0$, approaches μ^* when $\widetilde{\mathbf{R}}_q$ is applied repeatedly. Thus, the critical surface associated with μ^* is simply this sub-space.

4.6 Critical Exponents

For relating the properties of $\mathbf{R}_q$ with the critical exponents, consider the ferromagnetic critical point. The correlation lengths of the original system and the renormalized system are related as

$$\xi(\mu) = q\xi(\mu').$$

The parameter μ indicates the dependence of ξ on the parameters of H. First, consider the case of no magnetic field, i.e. $h = 0$. Then the parameters of H depend only on temperature and the dependence is analytic. For $\mu = \mu^*$, one gets

$$\xi(\mu^*) = q\xi(\mu^*).$$

Therefore, $\xi(\mu^*) = 0$ or ∞. That is, for the system defined by the hamiltonian corresponding to μ^*, the correlation length is either 0 or infinity. The latter case is of importance since the correlation length of the critical system is infinity.

Now, let $\mu = \mu_c$ be a point (near μ^*) on the critical surface. Then

$$\xi(\mu_c) = q\xi(\mu_c') = q\xi(\mathbf{R}_q\mu_c) = q\xi\Big(\mu^* + \widetilde{\mathbf{R}}_q\delta\mu_c\Big).$$

If a large value of q is taken, then $\widetilde{\mathbf{R}}_q\delta\mu_c \approx 0$. Then one finds that

$$\xi(\mu_c) = q\xi(\mu^*) = \infty.$$

Thus, systems represented by points on the critical surface also have infinite correlation lengths. This observation leads to a fundamental hypothesis that a critical system is represented by a point on the critical surface. For the ferromagnetic case, all the parameters represented by μ_c depend only on T_c since $h = 0$ on the critical surface.

Another hypothesis regarding the eigenvalues of $\widetilde{\mathbf{R}}_q$ is that there are only two eigenvalues, ρ_1 and ρ_h, which are greater than unity. That is, $y_1 > 0$, $y_h > 0$ and $y_j < 0$ for $j \geq 2$. This hypothesis is to be tested against RG calculations for ferromagnetic models. With this hypothesis, the exponents can be related to ρ_1 and the parameter a in $\alpha(q) = q^a$.

4.6.1 Correlation Length Exponent ν

The exponent ν, defined as $\xi \sim |T - T_c|^{-\nu}$, can be related to y_1 as follows. First of all, let $h = 0$. Consider a point μ close to μ^* but not on the critical surface. The deviation of μ from μ^* can be expressed as

$$\mu = \mu^* + \delta\mu = \mu^* + t_1\mathbf{e}_1 + \sum_{j\geq 2} t_j\mathbf{e}_j.$$

Application of the RG transformation l times yields a point $\mu^{(l)}$

$$\mu^{(l)} = \mu^* + \delta\mu^l = \mu^* + t_1 q^{ly_1}\mathbf{e}_1 + \sum_{j\geq 2} t_j q^{ly_j}\mathbf{e}_j.$$

For a sufficiently large value of l, this reduces to

$$\mu^{(l)} = \mu^* + t_1 q^{ly_1}\mathbf{e}_1 + O(q^{ly_2}),$$

since $y_j < 0$ for $j \geq 2$ and they can be ordered as $|y_2| \leq |y_3| \leq |y_4|$ etc. The correlation lengths of the systems corresponding to μ and $\mu^{(l)}$ are related by

$$\xi(\mu) = q^l \xi(\mu^{(l)}).$$

Then, for a large value of l, one gets

$$\xi(\mu) = q^l \xi\Big(\mu^* + t_1 q^{ly_1}\mathbf{e}_1 + O(q^{ly_2})\Big).$$

The initial point μ can be changed by changing temperature T. That is, the expansion coefficients t_j are functions of T. The components of μ, and hence $\{t_j\}$, are analytic functions of T. Note that $a_2 = a_2'(T - T_c)$. When $t_1 = 0,\ \mu$ is on the critical surface. Therefore, a Taylor expansion of t_1 around T_c yields

$$t_1(T) = A_1(T - T_c) + A_2(T - T_c)^2 + \cdots.$$

where A_1 etc. are constants. It is assumed that $A_1 \neq 0$. Thus for a large value of l and small $\Delta T = T - T_c$, $\xi(\mu)$ is given by

$$\xi(\mu) = q^l \xi\Big(\mu^* + A_1 \Delta T q^{ly_1}\mathbf{e}_1 + O(\Delta T)^2 + O(q^{ly_2})\Big).$$

This relation should be true for an arbitrary value of q and large l and hence in particular for $q^l = |\Delta T|^{-1/y_1}$. Thus

$$\xi(\mu) = |\Delta T|^{-1/y_1}\xi\Big(\mu^* \pm A_1\mathbf{e}_1 + O(\Delta T)^2 + O(\Delta T^{-y2/y_1})\Big),$$

which shows that the temperature dependence of ξ for small ΔT is like $|\Delta T|^{-\nu}$ with $\nu = 1/y_1$. Thus, the calculation of the eigenvalue $\rho_1(q)$ of the linearised RG transformation provides a method to obtain the exponent ν since they are related as

$$\nu = \frac{\ln(q)}{\ln(\rho_1)} = \frac{1}{y_1}.$$

4.6.2 Correlation Function Exponent η

There are four relations (scaling laws) among the six exponents and hence it is enough to calculate any two of them. Having found an expression for ν, a relation for η can be derived as follows. A factor $\alpha(q)$ was introduced to

adjust the magnitude of the Fourier amplitudes in the third step of RG. Using the multiplicative property of $\mathbf{R}_q$, this parameter was shown to have a power law dependence on q, namely, $\alpha(q) = q^a$. It was also stressed that the value of a has to be chosen so as to find an appropriate fixed point for $\mathbf{R}_q$. For the Gaussian model, it was shown that different fixed points can be obtained by giving different values to a. Let a point $\mu' = \widetilde{\mathbf{R}}_q\mu$, be generated starting from μ. The Fourier transform of the correlation functions of the two systems are related as

$$G(k,\mu) = q^{2a+d}G(qk, \widetilde{\mathbf{R}}_q\mu).$$

Iterating this equation l times, one obtains

$$G(k,\mu) = q^{l(2a+d)}G(q^l k, [\widetilde{\mathbf{R}}_q]^l\mu).$$

Let $\mu_c = \mu^* + \delta\mu_c$ be a point on the critical surface. Hence, this point corresponds to $h = 0$ and $T = T_c$. Since $\delta\mu_c$ has no projection on $\mathbf{e}_1$ and $\mathbf{e}_h$, one knows that

$$[\widetilde{\mathbf{R}}_q]^l\mu_c = \mu^* + \sum_{j\geq 2} t_j q^{ly_j}\mathbf{e}_j = \mu^* + O(q^{ly_2}),$$

for a large value of l. Therefore, $G(k,\mu_c)$ is given by

$$G(k,\mu_c) = q^{l(2a+d)}G\Big(q^l k, \mu^* + O(q^{ly_2})\Big),\ l \gg 1.$$

Since q is arbitrary, it can be chosen as $q^l = k^{-1}$ for small k. Since μ_c corresponds to T_c, one gets

$$G(k,T_c) = k^{-(2a+d)}G\Big(\pm 1, \mu^* + O(|k|^{-y_2})\Big),$$

for small k .But it is known that $G(k,T_c)$ varies as $k^{-2+\eta}$ where η is the correlation function exponent. Thus η and a are related as

$$a = \frac{1}{2}(2 - \eta - d).$$

Therefore, the value of a required to find an appropriate fixed point, yields η. The replacement of the Fourier amplitude $s_{i\mathbf{k}}$ in the second and third steps of RG can now be written as

$$s_{i\mathbf{k}} \rightarrow q^{a+d/2}s_{i\mathbf{k}'} = s^{1-\eta/2}s_{i\mathbf{k}'}\ , \mathbf{k}' = q\mathbf{k}.$$

Thus, a method to obtain two exponents can be devised within the framework of RG. What is remaining to be shown is that the scaling forms of correlation function and free energy density can also be derived within this framework.

4.7 Scaling Form of Correlation Function

To obtain the full scaling form of correlation function, it is necessary to include the field parameter h in μ. The functional equation for $G(k,\mu)$ is

$$G(k,\mu) = q^{l(2-\eta)} G(q^l k, [\widetilde{\mathbf{R}}_q]^l \mu),$$

where the relation $2a + d = 2 - \eta$ has been used. Now, consider a point μ close to μ^*, but not on the critical surface. Then μ can be written as

$$\mu = \mu^* + h\mathbf{e}_h + t_1\mathbf{e}_1 + \sum_{j\geq 2} t_j \mathbf{e}_j.$$

Repeated application (l times) of $\widetilde{\mathbf{R}}_q$ on μ yields

$$[\widetilde{\mathbf{R}}_q]^l \mu = \mu^* + hq^{ly_h}\mathbf{e}_h + t_1 q^{ly_1}\mathbf{e}_1 + O(q^{ly_2}),\ l \gg 1\ ,$$

where $y_h = a + d$. Therefore $G(k,\mu)$ satisfies

$$G(k,\mu) = q^{l(2-\eta)} G\Big(q^l k,\ \mu^* + hq^{ly_h}\mathbf{e}_h + t_1 q^{ly_1}\mathbf{e}_1 + O(q^{ly_2})\Big),$$

for a large value of l. Now, using

$$t_1(T) = A_1\Delta T + A_2 \Delta T^2 + \cdots,\ \Delta T = T - T_c,$$

and choosing $q^l = |\Delta T|^{-1/y_1}$ one obtains

$$\begin{aligned} G(k,\mu) &= |\Delta T|^{-(2-\eta)/y_1} G\Big(|\Delta T|^{-1/y_1} k, \mu^* + h|\Delta T|^{-y_h/y_1}\mathbf{e}_h \\ &\pm A_1\mathbf{e}_1 + O(\Delta T)^2 + O(|\Delta T|^{-y2/y_1})\Big), \end{aligned}$$

for small ΔT. Since $1/y_1 = \nu$, this is of the form

$$\begin{aligned} G(k,\mu) &= |\Delta T|^{-\nu(2-\eta)} G\Big(\frac{k}{|\Delta T|^\nu}, \mu^* + \frac{h}{|\Delta T|^\Delta}\mathbf{e}_h \pm A_1\mathbf{e}_1\Big) \\ &= k^{-2+\eta}\Big(\frac{k}{|\Delta T|^\nu}\Big)^{2-\eta} G\Big(\frac{k}{|\Delta T|^\nu}, \mu^* + \frac{h}{|\Delta T|^\Delta}\mathbf{e}_h \pm A_1\mathbf{e}_1\Big) \\ &= k^{-2+\eta} D_\pm\Big(\frac{k}{|\Delta T|^\nu}, \frac{h}{|\Delta T|^\Delta}\Big), \end{aligned}$$

where the $\pm$ sign stands for $\Delta T > 0$ and $\Delta T < 0$ respectively, and

$$\Delta = \frac{y_h}{y_1} = \frac{\nu}{2}(2 - \eta + d).$$

Thus the scaling form for correlation function emerges in a nice way.

4.8 Scaling Form of Free Energy Density

Before developing a detailed analysis leading to the scaling form of free energy density, consider a simplified derivation. The free energy densities of the original (represented by μ) and renormalized systems (represented by μ') can be related as

$$F(\mu) = q^{-d} F(\mu').$$

Such a relation was derived (with $d = 1$) for the 1-D Ising model. The factor q^{-d} arises from the change of volume $V' = Vq^{-d}$ accompanying spatial rescaling. The above relation is incomplete since an additive term, generated by the partial elimination of degrees of freedom, is not accounted in it. Omitting this contribution (see below), one obtains (as in the case of correlation function)

$$F(\mu) = q^{-ld} F\Big(\mu^* + hq^{ly_h}\mathbf{e}_h + t_1 q^{ly_1}\mathbf{e}_1 + O(q^{ly_2})\Big),\ l \gg 1.$$

Now choose $q^l = |\Delta T|^{-1/y_1}$ and use the expansion for t_1 to obtain

$$F(\mu) = |\Delta T|^{d/y_1} F\Big(\mu^* + \frac{h}{|\Delta T|^{\Delta}}\mathbf{e}_h \pm A_1\mathbf{e}_1 + O(\Delta T)^2 + O(|\Delta T|^{y2/y_1})\Big),$$

where $\Delta = y_h/y_1$. Writing $d/y_1 = 2 - \alpha$, or $\alpha = 2 - \nu d$, the scaling form for $F(\mu)$ reduces to

$$F(\mu) = |\Delta T|^{2-\alpha} Y_{\pm}\Big(\frac{h}{|\Delta T|^{\Delta}}\Big).$$

This derivation also yields the hyperscaling relation. However, it was noted earlier that the exponents of Gaussian approximation do not satisfy the hyperscaling law except for $d < 4$. This point will be discussed later. The other scaling relations can be obtained using the functional forms of F and G.

A more detailed derivation of the free energy density scaling ansatz is as follows. Since the additive constant term is unimportant in the original hamiltonian, let $a_0 = 0$. Then $H[s] = 0$ for $\mathbf{s} = 0$. But it is known that the RG transformation generates an additive constant to the new hamiltonian. This term arises as the contribution of the eliminated modes to the total free energy. Writing out this contribution explicitly, the RG transformation is

$$\exp\Big(-\frac{H'[\{s_{i\mathbf{k}'}\}]}{T} - A(\mu)\frac{L^d}{T}\Big) = \int \exp\Big(-\frac{H[\{s_{i\mathbf{k}}\}]}{T}\Big) \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}},$$

with $s_{i\mathbf{k}} \to s_{i\mathbf{k}'} q^{a+d/2}$. The constant $A(\mu)$ is the free energy density contributed by modes with k in Λ/q to Λ. In this definition, $H'[s] = 0$ when $\mathbf{s} = 0$. The free energy densities, $F(\mu)$ and $F(\mu')$, of the old and new hamiltonians are

defined as

$$\begin{aligned}\exp\Big(-\frac{F(\mu)}{T}L^d\Big) &= \int \exp(-\frac{H}{T}) \prod_{i\ k\le\Lambda} ds_{i\mathbf{k}}\ ,\\ \exp\Big(-\frac{F(\mu')}{T}L'^d\Big) &= \int \exp(-\frac{H'}{T}) \prod_{i\ k'\le\Lambda} ds_{i\mathbf{k}'}\ .\end{aligned}$$

Note that the additive constant generated by RG is to be separated so that the free energy density is the same function of the parameter sets μ or μ'. In the new hamiltonian, the volume factor is $L'^d = q^{-d}L^d$ because the unit of length is q times the old unit. The equation for $F(\mu)$ can be rewritten as

$$\exp\Big(-\frac{F(\mu)}{T}L^d\Big) = \int \prod_{i\ k\le\Lambda/q} ds_{i\mathbf{k}} \int \exp(-\frac{H}{T}) \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}}.$$

For modes with $k \le \Lambda/q$, the replacement $s_{i\mathbf{k}} \to s_{i\mathbf{k}'}q^{1-\eta/2}, \mathbf{k}' = q\mathbf{k}$ yields

$$\begin{aligned}&\exp\Big(-\frac{F(\mu)}{T}L^d\Big)\\ &= (\prod_{i\ k\le\Lambda/q} q^{1-\eta/2}) \int \prod_{i\ k'\le\Lambda} ds_{i\mathbf{k}'} \int \exp(-\frac{H}{T}) \prod_{i\ \Lambda/q<k\le\Lambda} ds_{i\mathbf{k}}\\ &\to (\prod_{i\ k\le\Lambda/q} q^{1-\eta/2}) \int \prod_{i\ k'\le\Lambda} ds_{i\mathbf{k}'} \exp\Big(-\frac{1}{T}H'[s_{i\mathbf{k}'}] - \frac{A(\mu)}{T}L^d\Big).\end{aligned}$$

In getting this relation, the definition of $H'[s_{i\mathbf{k}'}]/T$ has been used. Using the definition of $F(\mu')$, one gets

$$\begin{aligned}&\exp\Big(-\frac{F(\mu)}{T}L^d\Big)\\ &= \exp\Big[(1-\eta/2)\ln(q) \sum_{i\ k\le\Lambda/q} 1\Big] \exp\Big(-\frac{F(\mu')}{T}L'^d - \frac{A(\mu)}{T}L^d\Big).\end{aligned}$$

Thus $F(\mu)$ and $F(\mu')$ are related as

$$F(\mu) = q^{-d}F(\mu') + A(\mu) - \frac{T}{L^d}(1-\eta/2)\ln(q)n \sum_{k\le\Lambda/q} 1.$$

The last term has come due to the change of unit of length and the change in the magnitude of the remaining Fourier amplitudes. This, as well as the term $A(\mu)$, were not considered earlier. Since the density of points in k-space is $(L/2\pi)^d$,

$$\sum_{k\le\Lambda/q} 1 = (\frac{\Lambda}{q})^{-d}(\frac{L}{2\pi})^d.$$

Therefore,

$$F(\mu) = q^{-d}F(\mu') + X(\mu, q),$$

where the additive term $X(\mu, q)$ is

$$X(\mu, q) = A - T(1 - \eta/2)\ln(q)n\Lambda^{-d}(\frac{q}{2\pi})^d.$$

Note that $X(\mu, q) = 0$ for $q = 1$. Then, for $q \approx 1$ or $(q - 1)$ a small number, one can write

$$X(\mu, q) \approx [\frac{\partial X}{\partial s}]_{q=1}(q - 1) = X'(\mu)(q - 1).$$

Thus, the relation between $F(\mu)$ and $F(\mu')$ is

$$F(\mu) = q^{-d}F(\mathbf{R}_q\mu) + X'(\mu)(q - 1),\ q \approx 1.$$

Iteration of this equation l times yields

$$F(\mu) = q^{-ld}F([\mathbf{R}_q]^l\mu) + \sum_{m=0}^{l-1} q^{-md}X'([\mathbf{R}_q]^m\mu)(q - 1),\ q \approx 1. \tag{4.10}$$

For a large value of l, and hence a large value of $b = q^l$, but for q close to 1, the summation can be approximated by an integral. Putting $q^m = b'$ so that $q^{m+1} = b' + db'$ or $q^m(q - 1) = db'$, one finds

$$F(\mu) = b^{-d}F(\mathbf{R}_b\mu) + \int_1^b b'^{\ -d}X'(\mathbf{R}_{b'}\mu)\frac{db'}{b'}.$$

In this equation, the relation $(\mathbf{R}_q)^l\mu = \mathbf{R}_{q^l}\mu$ has been used. The integral can be split into two parts, one from 1 to b_0 and the other from b_0 to b. Then, assuming that $1 \ll b_0 \ll b$, $F(\mu)$ is given by

$$F(\mu) = b^{-d}F(\mathbf{R}_b\mu) + \int_1^{b_0}(\cdots) + \int_{b_0}^{b}(\cdots) = F_r + F_s,$$

where F_s (singular part) is defined as

$$F_s(\mu) = b^{-d}F(\mathbf{R}_b\mu) + \int_{b_0}^b b'^{\ -d}X'(\mathbf{R}_{b'}\mu)\frac{db'}{b'}.$$

The term F_r (regular part) is not expected to give any singular behaviour. When μ is close to μ^*, but not on the critical surface, $\mathbf{R}_b\mu$ (for large b) can be written as

$$\mathbf{R}_b\mu = \mu^* + hb^{y_h}\mathbf{e}_h + t_1b^{y_1}\mathbf{e}_1 + O(b^{y_2}).$$

Note that b' inside the integral in F_s is large since $1 \ll b_0$. Therefore,

$$\begin{aligned} F_s(\mu) &= b^{-d} F\Big(\mu^* + hb^{y_h}\mathbf{e}_h + t_1 b^{y_1}\mathbf{e}_1 + O(b^{y_2})\Big) \\ &+ \int_{b_0}^{b} b'^{\,-d} X'\Big(\mu^* + hb'^{\,y_h}\mathbf{e}_h + t_1 b'^{\,y_1}\mathbf{e}_1 + O(b'^{\,y_2})\Big)\frac{db'}{b'}. \end{aligned}$$

Now change the variable b' to bz. Then, the lower limit for z is $b_0/b \approx 0$ since b is large compared to b_0. Thus for large b, one gets

$$\begin{aligned} F_s(\mu) &= b^{-d} F\Big(\mu^* + hb^{y_h}\mathbf{e}_h + t_1 b^{y_1}\mathbf{e}_1\Big) \\ &+ b^{-d}\int_0^1 z^{-d} X'\Big(\mu^* + hb^{y_h} z^{y_h}\mathbf{e}_h + t_1 b^{y_1} z^{y_1}\mathbf{e}_1\Big)\frac{dz}{z}. \end{aligned}$$

Now, use the expansion, $t_1(T) \approx A_1\Delta T$, for T close to T_c, and further choose $b = |\Delta T|^{-1/y_1}$. Therefore $F_s(\mu)$ reduces to

$$\begin{aligned} F_s(\mu) &= |\Delta T|^{d/y_1} F\Big(\mu^* + \frac{h}{|\Delta T|^{y_h/y_1}}\mathbf{e}_h \pm A_1\mathbf{e}_1\Big) \\ &+ |\Delta T|^{d/y_1}\int_0^1 z^{-d} X'\Big(\mu^* + \frac{h}{|\Delta T|^{y_h/y_1}} z^{y_h}\mathbf{e}_h \pm A_1 z^{y_1}\mathbf{e}_1\Big)\frac{dz}{z}. \end{aligned}$$

This expression is clearly of the scaling form

$$F_s(\mu) = |\Delta T|^{2-\alpha} Y_{\pm}\Big(\frac{h}{|\Delta T|^{\Delta}}\Big),$$

where $2 - \alpha = d/y_1$ and $\Delta = y_h/y_1$.

The above analysis leading to the scaling form of free energy density shows two important points. First of all, it is found that the contribution $X(\mu, q)$ also has the scaling form when ΔT is small. Such a term does not arise in the derivation of the scaling form of correlation function since the contribution to $G(k)$ for small k arises purely from the long wavelength fluctuations while fluctuations at all wavelengths contribute to the free energy density. The second point is that only a part of the free energy density (which was called F_s) has the scaling behaviour.

4.8.1 Scaling Form in Finite Systems

In the previous chapter, a scaling form for the free energy density in a finite size system was introduced. This ansatz also can be derived using the above analysis. In a finite system Eq.(4.10) should be written as

$$F(\mu, L) = q^{-ld} F([\mathbf{R}_q]^l \mu, q^{-l} L) + \sum_{m=0}^{l-1} q^{-md} X'([\mathbf{R}_q]^m \mu)(q-1),\ q \approx 1,$$

where the dependence of F on the linear size L is indicated explicitly. The application of the RG transformation (l times) includes a spatial rescaling by a factor q^l and so the linear size of the renormalized system is $q^{-l}L$ as indicated on the right. The analysis leading to the scaling form for F_s can now be repeated to obtain

$$\begin{aligned} F_s(\mu, L) &= b^{-d} F\Big(\mu^* + hb^{y_h}\mathbf{e}_h + A_1 \Delta T b^{y_1}\mathbf{e}_1, b^{-1}L\Big) \\ &+ b^{-d}\int_0^1 z^{-d} X'\Big(\mu^* + hb^{y_h} z^{y_h}\mathbf{e}_h + A_1\Delta T b^{y_1} z^{y_1}\mathbf{e}_1\Big)\frac{dz}{z}, \end{aligned}$$

where $b = q^l$. In this expression, the size parameter L appears in the same way as the field and temperature variables and the exponent, -1, is similar to y_h and y_1. Now, the choice $b = |\Delta T|^{-1/y_1}$ yields the scaling form

$$F_s(\mu, L) = |\Delta T|^{2-\alpha} f_\pm\Big(\frac{h}{(\Delta T)^\Delta}, (\Delta T)^\nu L\Big),$$

which is exactly of the type hypothesized earlier. In the limit $L \to \infty$, the functions $f_\pm$ should reduce to $Y_\pm$ so that the infinite size scaling forms are recovered. The general RG procedures can not say anything more on the nature of these functions, they can be obtained only from detailed calculations on specific models.

4.9 Some Notes

A number of assumptions were introduced to extract the general features of critical phenomena using the RG transformation. (i) It has been implicitly assumed that a fixed point of the transformation exists and the transformation equations are analytic (so that they can be linearised) near the fixed point. (ii) The eigenfunctions of the linearised transformation are complete and hence can be used as a basis for the μ-space. (iii) There are only two eigenvalues (ρ_1 and ρ_h) of the linearised RG matrix, which exceed unity. (iv) In the expansion $t_1(\Delta T) = A_1\Delta T + O(\Delta T)^2$, the constant $A_1 \neq 0$.

In the representation in which $\widetilde{\mathbf{R}}_q$ is diagonal, the linearised RG transformation

$$\delta\mu' = \widetilde{\mathbf{R}}_q \delta\mu,$$

takes the form

$$t_j' = \rho_j t_j.$$

Now, if $\rho_j > 1$ (i.e. $y_j > 0$), t_j increases on repeated application of $\widetilde{\mathbf{R}}_q$ while it tends to zero if $\rho_j < 1$ (i.e. $y_j < 0$). If $\rho_j > 1$, the variable t_j is called a relevant variable. Similarly, if $\rho_j < 1$, it is called an irrelevant variable since

it does not play an important role near the fixed point. Those t_j for which $\rho_j = 1$ are known as marginal variables as they are unaffected by the RG transformation. The assumption that only ρ_1 and ρ_h are greater than unity, which leads to the scaling forms, makes t_1 and h relevant variables. Thus one may expect that the simple scaling forms for the correlation function and free energy density have to be modified if there are more relevant variables.

The scaling ansatz for the correlation function and free energy density were derived by considering repeated application (l times) of $\widetilde{\mathbf{R}}_q$ and then choosing $q^l \sim \xi$ and $q^l \sim k^{-1}$. The scaling forms resulted on assuming that terms like q^{ly_2} are negligible. Hence the region of temperature, in which the scaling forms are valid, is decided by the smallness of ξ^{y_2}, i.e. $|T - T_c|^{-\nu y_2}$. Similarly, the region of k, in which the scaling form of the correlation function is valid, is determined by the smallness of k^{-y_2}. Thus, scaling behaviour will be observed only when $|T - T_c|^{-\nu y_2}$ and k^{-y_2} are small quantities.

4.10 Universality of Critical Phenomena

Having gone through the deduction of scaling behaviour from the RG formalism, it is natural to see how the universality concepts of critical phenomena are built into it. It was shown that the eigenvalues ρ_1 and ρ_h of $\widetilde{\mathbf{R}}_q$ are related to the critical exponents. It is clear that different hamiltonians represented by points in the basin of attraction of a fixed point will approach it in a similar manner if the RG transformation is applied repeatedly. The points which do not lie on the critical surface associated with the fixed point, first of all, move closer to it as the irrelevant variables are reduced to negligible values. Thereafter, they move away from the fixed point. The manner in which these points move in the neighborhood of the fixed point (on applying the RG transformation) is dictated by the values of ρ_1 and ρ_h which in turn fix the exponents. Therefore, all systems represented by points in the basin of attraction of a fixed point will have the same critical exponents and hence will belong to the same universality class. It is possible that a particular model may yield different fixed points and their associated basins of attraction in the same parameter space. Then, systems corresponding to different fixed points will belong to different universality classes. The observed dependence of exponents on the order parameter dimension (n) and spatial dimension (d) suggests that there are different fixed points in the parameter space.

4.11 Gaussian Model - Exponents

The RG transformation equations for the Gaussian model have been derived earlier. Using the relation $a = (2 - \eta - d)/2$, they can be rewritten as

$$\begin{aligned} a_2' &= a_2 q^{2-\eta}, \\ c' &= c q^{-\eta}, \\ h' &= h q^{(2-\eta+d)/2}. \end{aligned}$$

One of the fixed points of the transformation is obtained by choosing $\eta = 0$. This is equivalent to the choice $a = (2-d)/2$ discussed earlier. Then the fixed point value of μ is

$$\mu^* = (a_2^*, c^*, h^*) = (0, c^*, 0).$$

Thus the fixed point hamiltonian is

$$\frac{H^*}{T} = c^* \int (\nabla s)^2 d\mathbf{x},$$

where c^* is an arbitrary constant. This fixed point is usually called the Gaussian fixed point. There is no need to linearise the transformations for the Gaussian model. Since the transformation matrix is diagonal, the eigenvalues (with $\eta = 0$) are

$$\begin{aligned} \rho_1(q) &= q^{y_1} = q^2, \\ \rho_2(q) &= q^{y_2} = 1, \\ \rho_h(q) &= q^{y_h} = q^{(2+d)/2}. \end{aligned}$$

Thus two exponents are $\eta = 0$ and $\nu = 1/y_1 = 1/2$. They are the same as those obtained from direct calculations. The scaling laws, now, provide the other exponents of the Gaussian model, and they are

$$\begin{array}{lll} \beta = & (\nu/2)(d - 2 + \eta) & = (d-2)/4 \\ \delta = & (d - \eta + 2)/(d + \eta - 2) & = (d+2)/(d-2) \\ \gamma = & \nu(2 - \eta) & = 1 \\ \alpha = & 2 - \nu\ d & = (4-d)/(2) \end{array}$$

Note that the exponent $\alpha = (4 - d)/2$ differs from the value $\alpha = 0$ for $d \geq 4$ obtained in the linearised L-G model. This point is discussed in next chapter.

4.12 Summary of RG Ideas

The RG approach has originated from an important observation that a system near its critical point has a large (in units of a basic length scale like lattice spacing) spatial correlation length. Therefore, two descriptions, differing in

small length scale features (or details), should be equivalent. From a given description (say, as provided by the Ising model), one may obtain an equivalent coarse grained description by a local reduction of degrees of freedom. Kadanoff proposed a method to implement such a reduction of degrees of freedom by averaging over cells of certain size. This approach was discussed in reference to the 1-D Ising model. However, Wilson's approach of integrating out short wavelength Fourier components turns out to yield a workable scheme for continuum models like the L-G model. This approach also has been discussed with reference to the Gaussian model.

Thus, starting from a given model (specified by a hamiltonian H), it may be possible to generate a sequence of models (represented by $H^{(l)}, l = 1, 2 \cdots$) by repeated application of the coarse graining operation. All these models are expected to be equivalent near the critical point since all of them contain the same long length scale details of the system. Thus, coarse graining operation appears to be a symmetry operation for describing critical phenomena. The equivalence of the generated models may be shown up in the almost similar forms of the hamiltonians $H^{(l)}$. The initial hamiltonian H contains certain parameters and its form should be sufficiently general so that the generated hamiltonians $H^{(l)}$ also have the same form.

A particular hamiltonian can be characterized in terms of the values of its parameters, and therefore one can imagine the hamiltonian to be represented by a point in the parameter space. Coarse graining changes the values of the parameters, and thus, leads to a new point in the parameter space and a new hamiltonian. The effect of coarse graining can be extracted by comparing the parameters of the two hamiltonians. However, parameters in a hamiltonian are characteristic of the shortest length scale of that description. Since coarse graining changes the shortest length scale, it is necessary to alter the length unit so that the coarse grained description also has the same numerical value of the shortest length scale. Having done this, it is possible to compare the parameter sets of the two descriptions. Thus the coarse graining operation and spatial length rescaling are the two important steps in the RG transformation. When these steps are pictured as a transformation in the parameter space, it is natural to look for the fixed points of the transformation. The systems represented by the fixed points are invariant under the RG transformation. Critical systems with large correlation lengths are expected to be invariant under coarse graining and so it is natural to see if they can be associated with the fixed points. Alternatively, one may say that if a fixed point can not be identified, the idea of coarse graining may not be fruitful. For the Gaussian model, it was seen that a renormalization of the remaining degrees of freedom is necessary for identifying the proper fixed point. Thus the RG transformation contains three steps, (i) a partial reduction of degrees of freedom, (ii)

spatial length rescaling, and (iii) a renormalization of the remaining degrees of freedom.

The relation connecting the correlation lengths of equivalent systems generated via RG has shown that the systems represented by the fixed points have infinite correlation lengths. All systems lying on the critical surface associated with a fixed point also have infinite correlation lengths. The critical surface was defined as the set of all points in the parameter space which approach the fixed point by repeated RG transformation. It was also argued that in the vicinity of a fixed point, the RG transformation can be linearised and represented by a linear operator. Assuming that the eigenvalues are discrete, the eigenfunctions are complete, and only two eigenvalues are greater than unity, it was possible to relate these eigenvalues to the critical exponents. It was also possible to derive the scaling forms and hence the scaling relations among the exponents. This picture was also able to account for the universality observed in critical phenomena.

The RG approach does not say anything about the existence of fixed points associated with any model. This can be explored only by studying the model explicitly. So, what is remaining is to study the models and see if the above picture applies or not. This has already been done for the 1-D Ising model and the Gaussian model which can also be solved exactly with out invoking RG ideas. Nontrivial models (like the L-G model) require approximate methods for implementing the first step in RG approach. Some of these methods are developed in the following chapters.

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Chapter 5

Wilson's Recursion formulae

A nontrivial application of the RG ideas to the L-G model is developed in the present chapter. This was the first application, by Wilson, which led to a calculation of critical exponents. For simplicity, the calculations are restricted to the case of a one component ($n = 1$) order parameter. A generalized L-G form for H/T is

$$\frac{H[s]}{T} = \int_V \Big[U(s) + c\{\nabla s(\mathbf{x})\}^2\Big] dx,$$

where $U(s)$ is a function of s^2. The magnetic field term is not included in $U(s)$ since y_h and, hence, $\Delta = y_h/y_1$ depend only on η and d. $U(s)$ may be expanded as

$$U(s) = \sum_{m=1}^{\infty} a_{2m} s^{2\ m}.$$

The first two terms yield the usual L-G model. The general form of $U(s)$ is necessary since it will be shown that, starting with the L-G form, all even powers of $s(\mathbf{x})$ are generated by the first step of RG. Thus, this example will show the need to consider a large parameter space, in the present case $\mu = (c, a_2, a_4, \cdots)$.

As mentioned earlier, there is a characteristic length scale b in the spatial variation of $s(\mathbf{x})$. This fact is expressed by imposing a cut-off wave vector $\Lambda = 2\pi/b$ in the Fourier expansion

$$s(\mathbf{x}) = \frac{1}{L^{d/2}} \sum_{k \le \Lambda} \exp(\imath \mathbf{k} \cdot \mathbf{x}) s_{\mathbf{k}}.$$

It is known that when $H[s]$ is expressed in terms of $s_{\mathbf{k}}$, the quartic and higher order terms give rise to very complicated terms. Wilson's recursion formulae are derived by treating these terms in an approximate way. First of all $s(\mathbf{x})$ is written as

$$s(\mathbf{x}) = s'(\mathbf{x}) + \phi(\mathbf{x}),$$

where $s'(\mathbf{x})$ and $\phi(\mathbf{x})$ are given by

$$\begin{aligned} s'(\mathbf{x}) &= \frac{1}{L^{d/2}} \sum_{k \leq \Lambda/q} \exp(\imath \mathbf{k} \cdot \mathbf{x}) s_{\mathbf{k}}, \\ \phi(\mathbf{x}) &= \frac{1}{L^{d/2}} \sum_{\Lambda/q < k \leq \Lambda} \exp(\imath \mathbf{k} \cdot \mathbf{x}) s_{\mathbf{k}}. \end{aligned}$$

Thus $s'(\mathbf{x})$ contains modes with smaller wave vectors while $\phi(\mathbf{x})$ contains larger wave vectors. The first step in implementing RG is to obtain $H''[s']$ by integrating out the Fourier components in $\phi(\mathbf{x})$. Thus $H''[s']$ is written as a functional integral

$$\exp\Big(\frac{H''[s']}{T} - A\frac{L^d}{T}\Big) = \int D\phi \exp\Big(-\frac{H[s'+\phi]}{T}\Big),$$

where A is the contribution to free energy density from $\phi(\mathbf{x})$. The functional integration is a notation which implies that contributions from all possible $\phi(\mathbf{x})$ have to be added up to obtain the new hamiltonian $H''[s']$. One of the usual ways to achieve this is to express $H[s'+\phi]/T$ in terms of the Fourier components of $\phi(\mathbf{x})$ and then integrate over the Fourier amplitudes. Since the representation of $\phi(\mathbf{x})$ in the Fourier basis is not convenient, a different method to effect the functional integration has to be attempted. Note that

$$\begin{aligned} H[s'+\phi] &= \int_V \Big[U(s'+\phi) + c\{\nabla s'(\mathbf{x}) + \nabla\phi(\mathbf{x})\}^2\Big] d\mathbf{x} \\ &= \int_V \Big[U(s'+\phi) + c\{\nabla s'(\mathbf{x})\}^2 + c\{\nabla\phi(\mathbf{x})\}^2\Big] d\mathbf{x}. \end{aligned}$$

The cross term does not contribute since

$$\int_V \nabla s'(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}) d\mathbf{x} = \sum_{k \leq \Lambda/q} \sum_{\Lambda/q < k' \leq \Lambda} \mathbf{k} \cdot \mathbf{k}' s_{\mathbf{k}} s_{\mathbf{k}'} \delta(\mathbf{k} - \mathbf{k}') = 0,$$

because the ranges of $\mathbf{k}$ and $\mathbf{k}'$ do not overlap.

5.1 Wilson's Functions

To carry out the functional integration, Wilson introduced a new basis set of functions. The Fourier expansion uses the basis functions

$$v_{\mathbf{k}} = \frac{1}{L^{d/2}} \exp(\imath \mathbf{k} \cdot \mathbf{x}),$$

however, they are not convenient. Let $w(\mathbf{x})$ be the most localized function, around $\mathbf{x} = 0$, that can be constructed by superposing the Fourier basis $v_{\mathbf{k}}$

with $\Lambda/q < k \leq \Lambda$. That is

$$w(\mathbf{x}) = \frac{1}{L^{d/2}} \sum_{\Lambda/q<k\leq\Lambda} \exp(\imath \mathbf{k}\cdot\mathbf{x}) a_{\mathbf{k}}.$$

The coefficients $a_{\mathbf{k}}$ are chosen such that $w(\mathbf{x})$ is the most localized function around $\mathbf{x} = 0$. The wave vectors in $w(\mathbf{x})$ span the volume

$$V_k = (2\Lambda)^d - \frac{(2\Lambda)^d}{q^d} = (2\Lambda)^d(1 - q^{-d}),$$

in the k-space. The minimum volume Ω that $w(\mathbf{x})$ spans in the coordinate space is restricted by the condition $\Omega V_k \geq (2\pi)^d$. Therefore

$$\Delta x = \Omega^{1/d} = \frac{2\pi}{V_k^{1/d}},$$

is the linear dimension of the region, around $\mathbf{x} = 0$, in which $w(\mathbf{x})$ is significant. With $\mathbf{k}$ restricted over a finite region, it is not possible to make $w(\mathbf{x})$ vanish outside Ω. The assumption is that $a_{\mathbf{k}}$'s can be chosen such that $w(\mathbf{x})$ is negligible outside Ω. Some important properties of the functions $\{w(\mathbf{x})\}$ are the following.

(i) First of all note that

$$\int_V w(\mathbf{x})d\mathbf{x} = \int_\Omega w(\mathbf{x})d\mathbf{x} = 0,$$

since $w(\mathbf{x})$ does note contain the $\mathbf{k} = 0$ mode.

(ii) Imagine a lattice with spacing Δx. Then, consider the set of functions $w_l(\mathbf{x}) = w(\mathbf{x}-\mathbf{x}_l)$ where $\mathbf{x}_l$'s are the lattice points with spacing Δx. Therefore $w_l(\mathbf{x})$ can be written as

$$w_l(\mathbf{x}) = \frac{1}{L^{d/2}} \sum_{\Lambda/q<k\leq\Lambda} \exp\{\imath \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_l)\} a_{\mathbf{k}}.$$

By assumption, $w_l(\mathbf{x})$ and $w_{l'}(\mathbf{x})$ $(l \neq l')$ do not overlap. Therefore they are orthogonal, and so

$$\int_V w_l(\mathbf{x}) w_{l'}(\mathbf{x}) d\mathbf{x} = 0, \ \ for\ l \neq l'.$$

The magnitude of $w_l(\mathbf{x})$) can be normalized (by adjusting the values of $\{a_{\mathbf{k}}\}$ by a constant) in any case. So the orthogonality condition is

$$\int_V w_l(\mathbf{x}) w_{l'}(\mathbf{x}) d\mathbf{x} = \delta_{ll'}.$$

In terms of Fourier amplitudes, this means that

$$\sum_{\Lambda/q<k\leq\Lambda} \exp\{\imath\mathbf{k}\cdot(\mathbf{x}_l-\mathbf{x}_{l'})\}|a_{\mathbf{k}}|^2=\delta_{ll'}.$$

(*iii*) It is further assumed that the functions $\{\nabla w_l(\mathbf{x})\}$ are also orthogonal. That is,

$$\begin{aligned}&\int_V \nabla w_l(\mathbf{x})\cdot\nabla w_{l'}(\mathbf{x})d\mathbf{x}\\ &= \sum_{\Lambda/q<k\leq\Lambda} a_{\mathbf{k}}\imath\mathbf{k}\exp(\imath\mathbf{k}\cdot\mathbf{x})\cdot\sum_{\Lambda/q<k'\leq\Lambda} a_{\mathbf{k}}\imath\mathbf{k}'\exp(\imath\mathbf{k}'\cdot\mathbf{x})\\ &\times\ \frac{1}{L^d}\int\exp\{\imath(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}\}d\mathbf{x}=\sum_{\Lambda/q<k\leq\Lambda} k^2|a_{\mathbf{k}}|^2\exp\{\imath\mathbf{k}\cdot(\mathbf{x}_l-\mathbf{x}_{l'})\}.\end{aligned}$$

Assuming that $q\approx 1$, and hence k^2 can be replaced by a mean value k_m^2, one gets

$$\begin{aligned}&\int_V \nabla w_l(\mathbf{x})\cdot\nabla w_{l'}(\mathbf{x})d\mathbf{x}\\ &\approx\ k_m^2\sum_{\Lambda/q<k\leq\Lambda}|a_{\mathbf{k}}|^2\exp\{\imath\mathbf{k}\cdot(\mathbf{x}_l-\mathbf{x}_{l'})\}=k_m^2\delta_{ll'}.\end{aligned}$$

Thus, for the orthogonality of $\{\nabla w_l(\mathbf{x})\}$, it is necessary that $q\approx 1$.

(*iv*) Another assumption on $w_l(\mathbf{x})$ is that $|w_l(\mathbf{x})|$ is spatially constant in the region Ω surrounding the point $\mathbf{x}_l$. Since

$$\int_\Omega w_l(\mathbf{x})d\mathbf{x}=0,$$

this means that

$$w_l(\mathbf{x})=\pm\,|w_l(\mathbf{x})|,$$

where the + sign is for one half of Ω and the − sign is for the remaining half of Ω. Together with this assumption, the normalization condition on $w_l(\mathbf{x})$ yields the result

$$|w_l(\mathbf{x})|^2\Omega=1\ \textit{or}\ \ |w_l(\mathbf{x})|=\frac{1}{\sqrt{\Omega}}.$$

Thus one gets

$$w_l(\mathbf{x})=\pm\,\frac{1}{\sqrt{\Omega}},$$

for the two half of Ω respectively. The variation of $w_l(\mathbf{x})$ is shown in Figure 5.1.

(*v*) The function $\phi(\mathbf{x})$ can be expressed as a superposition of $w_l(\mathbf{x})$. That is,

$$\phi(\mathbf{x})=\sum_l \phi_l w_l(\mathbf{x}),$$

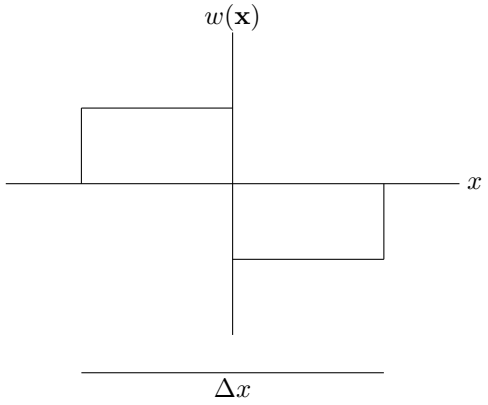

Figure 5.1: Wilson Function.

where $\{\phi_l\}$ are the combining coefficients.

With the assumptions listed above, it is possible to carry out the functional integration over $\phi(\mathbf{x})$ and obtain $H''[s']$. Using assumption (iii), one gets

$$\int_V [\nabla\phi(\mathbf{x})]^2 d\mathbf{x}$$
$$= \sum_l \sum_{l'} \phi_l \phi_{l'} \int_V \nabla w_l(\mathbf{x}) \cdot \nabla w_{l'}(\mathbf{x}) d\mathbf{x} = \sum_l \phi_l^2 k_m^2,$$

where k_m^2 is the mean value of k^2 (in the interval Λ/q to Λ) introduced earlier. Thus, the implicit assumption, in obtaining this result, is that $q \approx 1$. Now, consider the term

$$\int_V U(s'+\phi) d\mathbf{x} = \sum_l \int_\Omega U(s'+\phi) d\mathbf{x}.$$

According to assumption (ii), $w_l(\mathbf{x})$ does not overlap with $w_{l'}(\mathbf{x})$ when $\mathbf{x}$ is in Ω. Therefore,

$$\int_\Omega U(s'+\phi) d\mathbf{x} \approx \sum_l \int_\Omega U[s'(\mathbf{x}) + \phi_l w_l(\mathbf{x})] d\mathbf{x}.$$

To simplify this relation further, $s'(\mathbf{x})$ may be taken to be slowly varying in the region Ω. That means $s'(\mathbf{x}) \approx s'(\mathbf{x}_l)$ for $\mathbf{x}$ in Ω. Then, use of assumption (iv) yields

$$\int_\Omega U(s'+\phi) d\mathbf{x} \approx \sum_l \frac{\Omega}{2} \Big(U\Big[s'(\mathbf{x}_l) + \frac{\phi_l}{\sqrt{\Omega}}\Big] + U\Big[s'(\mathbf{x}_l) - \frac{\phi_l}{\sqrt{\Omega}}\Big] \Big). \tag{5.1}$$

Therefore, Eqs(5.1) and (5.2) give

$$\begin{aligned}\int_V \Big[c\{\nabla\phi(\mathbf{x})\}^2 + U(s'+\phi)\Big]d\mathbf{x} \\ = \sum_l c\phi_l^2 k_m^2 + \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) + \frac{\phi_l}{\sqrt{\Omega}}\Big] + \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) - \frac{\phi_l}{\sqrt{\Omega}}\Big].\end{aligned}$$

The functional integral over $\phi(\mathbf{x})$ can now be replaced by a multiple integral over the amplitudes $\{\phi_l\}$.

5.2 Recursion Formulae

The new hamiltonian $H''[s']$ can now be written as

$$\exp\Big(\frac{H''[s']}{T} - \frac{AL^d}{T}\Big) = \exp\Big(-c\int_V \{\nabla s'(\mathbf{x})\}^2 d\mathbf{x}\Big)$$

$$\times \int \prod_l d\phi_l \exp\Big(-\sum_l c\phi_l^2 k_m^2 + \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) + \frac{\phi_l}{\sqrt{\Omega}}\Big] + \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) - \frac{\phi_l}{\sqrt{\Omega}}\Big]\Big).$$

where the integrals over ϕ_l run over $-\infty$ to ∞. Defining the integral

$$I[s'(\mathbf{x}_l)] = \int_{-\infty}^{\infty} d\phi_l \exp\Big(-c\phi_l^2 k_m^2 - \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) + \frac{\phi_l}{\sqrt{\Omega}}\Big] - \frac{\Omega}{2}U\Big[s'(\mathbf{x}_l) - \frac{\phi_l}{\sqrt{\Omega}}\Big]\Big),$$

$H''[s']$ can be expressed as

$$\begin{aligned}\exp\Big(\frac{H''[s']}{T} - \frac{AL^d}{T}\Big) &= \exp\Big[-c\int_V \{\nabla s'(\mathbf{x})\}^2 d\mathbf{x}\Big] \prod_l I[s'(\mathbf{x}_l)] \\ &= \exp\Big[-c\int_V \{\nabla s'(\mathbf{x})\}^2 d\mathbf{x} + \sum_l \ln\{I[s'(\mathbf{x}_l)]\}\Big] \\ &= \exp\Big[-c\int_V \{\nabla s'(\mathbf{x})\}^2 d\mathbf{x} + \Omega^{-1}\int_V \ln\{I[s'(\mathbf{x})]\}d\mathbf{x}\Big],\end{aligned}$$

where the summation in the last term has been replaced by a volume integral. The last expression gives

$$\begin{aligned}\frac{H''[s']}{T} - \frac{AL^d}{T} &= \int_V \Big[c\{\nabla s'(\mathbf{x})\}^2 - \Omega^{-1}\ln\{I[s'(\mathbf{x})]\}\Big]d\mathbf{x} \\ &= \int_V \Big[c\{\nabla s'(\mathbf{x})\}^2 - \frac{1}{\Omega}\ln\Big\{\frac{I[s'(\mathbf{x})]}{I(0)}\Big\}\Big]d\mathbf{x} - \frac{V}{\Omega}\ln\{I(0)\}.\end{aligned}$$

Since the hamiltonian must be zero when $s' = 0$, H''/T can now be identified as

$$\frac{H''[s']}{T} = \int_V \Big[c\{\nabla s'(\mathbf{x})\}^2 d\mathbf{x} - \frac{1}{\Omega}\ln\Big\{\frac{I[s'(\mathbf{x})]}{I(0)}\Big\}\Big] d\mathbf{x}.$$

The free energy density term is

$$\frac{A}{T} = \frac{1}{\Omega}\ln\{I(0)\}.$$

Thus, starting with the hamiltonian

$$\frac{H[s]}{T} = \int_V [U(s) + c\{\nabla s(\mathbf{x})\}^2] d\mathbf{x},$$

the new hamiltonian obtained is

$$\frac{H''[s']}{T} = \int_V [U''(s') + c\{\nabla s'(\mathbf{x})\}^2] d\mathbf{x},$$

where $U''(s')$ is given by

$$U''(s') = -\frac{1}{\Omega}\ln\Big\{\frac{I(s')}{I(0)}\Big\}.$$

The integral $I(s')$ can be rewritten as

$$I(s') = \int_{-\infty}^{\infty} d\phi \exp\Big(-c\phi^2 k_m^2 - \frac{\Omega}{2}\Big\{U\Big[s' + \frac{\phi}{\sqrt{\Omega}}\Big] + U\Big[s' - \frac{\phi}{\sqrt{\Omega}}\Big]\Big\}\Big).$$

The factor Ω (which is rather arbitrary) can be removed by the definitions

$$U(s) = \frac{1}{\Omega}Q(s), U''(s') = \frac{1}{\Omega}Q''(s').$$

Then the two hamiltonians are given by

$$\begin{aligned}\frac{H[s]}{T} &= \int_V \Big[\frac{1}{\Omega}Q(s) + c\{\nabla s(\mathbf{x})\}^2\Big] d\mathbf{x},\\ \frac{H''[s']}{T} &= \int_V \Big[\frac{1}{\Omega}Q''(s') + c\{\nabla s'(\mathbf{x})\}^2\Big] d\mathbf{x},\end{aligned}$$

where

$$Q''(s') = -\ln\Big[\frac{I(s')}{I(0)}\Big].$$

The integral $I(s')$ reduces to

$$I(s') = \int_{-\infty}^{\infty} d\phi \exp\Big[-ck_m^2\Omega\phi^2 - \frac{1}{2}\{Q(s'+\phi) + Q(s'-\phi)\}\Big]. \tag{5.2}$$

Thus the first step of RG transformation is completed. Now, the unit of length has to be changed by a factor q and the scale factor $\alpha(q)$ for changing the magnitude of $s'(\mathbf{x})$ is to be introduced. These changes lead to the following replacements.

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}' &&= \frac{\mathbf{x}}{q}, \\ s'(\mathbf{x}) &\to && \alpha(q)s(\mathbf{x}'), \\ \nabla_{\mathbf{x}} &\to && \frac{1}{q}\nabla_{\mathbf{x}'}, \\ \int_V (\cdots)d\mathbf{x} &\to && q^d \int_{V'} (\cdots)d\mathbf{x}'. \end{aligned}$$

The factor $\alpha(q) = q^a$ and it is known that a and the exponent η are related as

$$a = 1 - \frac{\eta}{2} - \frac{d}{2}.$$

So the hamiltonian $H''[s']$ is to be replaced by

$$\begin{aligned} \frac{H'[s]}{T} &= \int_{V'} \Big[\frac{1}{\Omega}Q''\{\alpha(q)s(\mathbf{x}')\} + \frac{c}{q^2}\nabla\{\alpha(q)s(\mathbf{x}')\}^2\Big]q^d d\mathbf{x}' \\ &= \int_{V'} \Big[\frac{1}{\Omega}Q'\{s(\mathbf{x}')\} + c'\{\nabla s(\mathbf{x}')\}^2\Big]d\mathbf{x}', \end{aligned}$$

where the new variables defined are

$$\begin{aligned} c' &= \frac{c}{q^2}\alpha^2(q)q^d = cq^{-\eta}, \\ Q'(s) &= q^d Q''\Big[\alpha(q)s\Big] = q^d Q''\Big[q^{1-\eta/2-d/2}s\Big] = -q^d \ln\Big[\frac{I\{q^{1-\eta/2-d/2}s\}}{I(0)}\Big]. \end{aligned}$$

$I(s)$ and $Q(s)$ are related as in Eq.(5.3). These equations thus define the RG transformation. The value of η (or equivalently a) has to be chosen so as to identify the appropriate fixed point. Note that if $\eta \neq 0$, then repeated application of the transformation leads to $c = 0$. In other words, the fixed point value of c, defined as $c^* = c^* q^{-\eta}$ is zero if $\eta \neq 0$. When $c^* = 0$, the fixed point hamiltonian does not contain the gradient term and therefore it can not show any spatial correlation between the spin values. Such a hamiltonian can not be associated with a critical system. Hence η should be taken as zero. The approximations introduced to derive the RG transformation force the value of η to be zero. Thus the transformation of the other parameters (in $Q(s)$ or $U(s)$) are given by

$$Q'(s) = -q^d \ln\Big[\frac{I(q^{1-d/2}s)}{I(0)}\Big],$$

$$I(s) = \int_{-\infty}^{\infty} d\phi \exp\Big[- z\phi^2 - \frac{1}{2}\{Q(s+\phi) + Q(s-\phi)\}\Big].$$

where $z = ck_m^2\Omega$ is a function of q since k_m^2 as well as Ω are functions of q. If $Q(s)$ is expressed as a power series in s^2, then the above equations define the transformation for the coefficients of the power series. Since there are an infinite number of coefficients in the power series, the above equations show an example of RG transformation in an infinite dimensional parameter space.

5.2.1 Approximations

Two main approximations have been introduced in deriving the recursion formula. The first one is that the functions $\{\nabla w_l(\mathbf{x})\}$ are orthogonal. For this to hold, the variation of k^2 in the interval Λ/q to Λ must be small so that it can be replaced by a mean value. This implies that $1 \gg (\Lambda - \Lambda/q)/(\Lambda/q)$ which shows that $1 \gg q - 1$ or $2 \gg q$. The second assumption is that $s'(\mathbf{x})$ is practically constant in Ω over which $w_l(\mathbf{x})$ is significant. The wave vectors in $s'(\mathbf{x})$ lie in the interval 0 to Λ/q. So the smallest wavelength in $s'(\mathbf{x})$ is

$$\lambda_{min} = \frac{2\pi}{k_{max}} = \frac{2\pi q}{\Lambda} = bq.$$

For the variation of $s'(\mathbf{x})$ in Ω to be negligible, this wavelength should be large compared to the linear size of Ω. Therefore, this condition may be expressed as

$$\frac{\lambda_{min}}{2} \gg \Omega^{1/d}.$$

Thus one gets another condition

$$\frac{bq}{2} \gg \frac{b}{2}(1 - q^{-d})^{-1/d},$$

which means

$$(q^d - 1)^{1/d} \gg 1 \; or \; q^d \gg 2.$$

Thus there are conflicting requirements on the scale parameter q. Also note that the approximation, $s'(\mathbf{x})$ is constant in Ω, is the reason which forces the exponent η to be zero. An improved recursion formula which relaxes this assumption has been derived by Golner and it yields a nonzero value for η.

5.2.2 Numerical Calculations

Wilson performed numerical calculations with the recursion formula for $d = 3$. Choosing a value of $q = 2$ (a compromise value), the fixed point function $Q^*(s)$

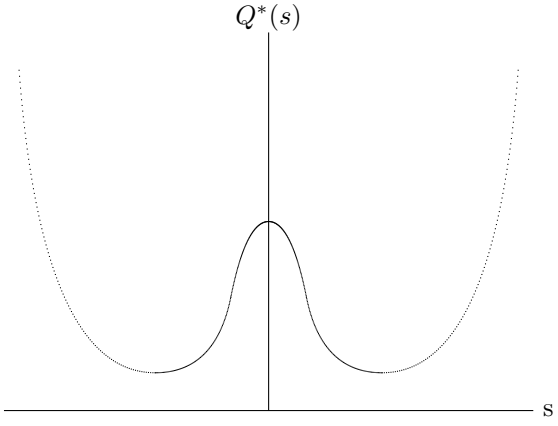

Figure 5.2: Fixed Point Function.

is determined by solving the non-linear integral equation

$$\begin{aligned} Q^*(s) &= -2^d \ln\Big[\frac{I^*(2^{1-d/2}s)}{I(0)}\Big], \\ I^*(s) &= \int_{-\infty}^{\infty} d\phi \exp\Big[-z\phi^2 - \frac{1}{2}\Big\{Q^*(s+\phi) + Q^*(s-\phi)\Big\}\Big]. \end{aligned}$$

The parameter $z = ck_m^2\Omega$ was chosen to be unity since c is arbitrary. The fixed point function has a behavior shown in Figure 5.2. Once $Q^*(s)$ is obtained, the transformation equation can be linearised by writing

$$\begin{aligned} Q'(s) &= Q^*(s) + \delta Q'(s) \\ Q(s) &= Q^*(s) + \delta Q(s). \end{aligned}$$

Thus one finds

$$\begin{aligned} &Q^*(s) + \delta Q'(s) \\ &= -2^d \ln\Big[I^*(2^{1-d/2}s) + \delta I(2^{1-d/2}s)\Big] - 2^d \ln\Big[I^*(0) + \delta I(0)\Big], \end{aligned}$$

where δI is the change in I when Q is changed from Q^* to $Q^* + \delta Q$. For small δQ, δI is small and therefore

$$\ln(I^* + \delta I) \approx \ln(I^*) + \ln\Big(1 + \frac{\delta I}{I^*}\Big) \approx \ln(I^*) + \frac{\delta I}{I^*}.$$

Thus $\delta Q'(s)$ is given by

$$\delta Q'(s) = -2^d\Big[\frac{\delta I(2^{1-d/2}s)}{I^*(2^{1-d/2}s)} - \frac{\delta I(0)}{I^*(0)}\Big].$$

From the definition of $I(s)$ (with z=1), one gets

$$\begin{aligned} I^*(s)+\delta I(s) &= \int_{-\infty}^{\infty} d\phi \exp\Big[-\phi^2-\frac{1}{2}\{Q^*(s+\phi)+Q^*(s-\phi)\}\Big] \\ &\qquad \times\Big[1-\frac{1}{2}\{\delta Q(s+\phi)+\delta Q(s-\phi)\}\Big]. \end{aligned}$$

Therefore $\delta I(s)$ is given by

$$\begin{aligned} \delta I(s) &= -\frac{1}{2}\int_{-\infty}^{\infty} d\phi \exp\Big[-\phi^2-\frac{1}{2}\{Q^*(s+\phi)+Q^*(s-\phi)\}\Big] \\ &\qquad \times\Big[\delta Q(s+\phi)+\delta Q(s-\phi)\Big] \\ &= -\int_{-\infty}^{\infty} d\phi \exp\Big[-\phi^2-\frac{1}{2}\{Q^*(s+\phi)+Q^*(s-\phi)\}\Big]\delta Q(s+\phi) \\ &= -\int_{-\infty}^{\infty} d\phi \exp\Big[-(\phi-s)^2-\frac{1}{2}\{Q^*(\phi)+Q^*(2s-\phi)\}\Big]\delta Q(\phi). \end{aligned}$$

The linearised transformation equation then becomes

$$\begin{aligned} &\delta Q'(s) \\ &= \frac{2^d}{I^*(w\ s)}\int_{-\infty}^{\infty} d\phi \exp\Big[-(\phi-w\ s)^2-\frac{1}{2}\{Q^*(\phi)+Q^*(w\ s-\phi)\}\delta Q(\phi)\Big] \\ &- \frac{2^d}{I^*(0)}\int_{-\infty}^{\infty} d\phi \exp\Big[-\phi^2-\frac{1}{2}\{Q^*(\phi)+Q^*(-\phi)\}\Big]\delta Q(\phi), \end{aligned}$$

where $w = 2^{1-d/2}$. This equation can be rewritten as

$$\delta Q'(s) = \int_{-\infty}^{\infty} d\phi\ T(s,\phi)\ \delta Q(\phi),$$

where the kernel $T(s,\phi)$ is given by

$$\begin{aligned} &T(s,\phi) \\ &= \frac{2^d}{I^*(w\ s)}\exp\Big[-(\phi-w\ s)^2-\frac{1}{2}\{Q^*(\phi)+Q^*(w\ s-\phi)\}\Big] \\ &- \frac{2^d}{I^*(0)}\exp\Big[-\phi^2-\frac{1}{2}\{Q^*(\phi)+Q^*(-\phi)\}\Big]\delta Q(\phi). \end{aligned}$$

This linear transformation is analogous to the linearised RG matrix $\widetilde{\mathbf{R}}_q$. For the RG picture to hold, this kernel should have eigenvalues $\rho_1 > 1$ and $\rho_j < 1$

for $j \geq 2$. The eigenvalues can be determined numerically and then the correlation length exponent ν can be obtained as $\rho_1 = 2^{1/\nu}$. For $d = 3$, Wilson thus obtained $\nu = 0.609$. The scaling law $\gamma = (2 - \eta)\nu$ then yields $\gamma = 1.218$ (since η has already been obtained as 0). These results may be compared with $\nu = 0.636$ and $\gamma = 1.25$ obtained using high temperature series expansion of the partition function. Wilson's recursion formulae can be extended to the case of multi-component order parameter. Calculations similar to that described show that $\gamma = 1.29$ for $n = 2$ and $\gamma = 1.36$ for $n = 3$. The high temperature series results are $\gamma = 1.32 \pm 0.01$ for $n = 2$ and 1.38 ± 0.01 for $n = 3$. Some analytical results that can be obtained for the Gaussian model and a perturbed Gaussian model are discussed below.

5.3 Gaussian Model via Recursion formulae

The Gaussian model is obtained by taking the $m = 1$ term in the power series expansion of $Q(s)$. That is

$$Q(s) = a_2 s^2.$$

Then $I(s)$ becomes

$$\begin{aligned} I(s) &= \int_{-\infty}^{\infty} d\phi \exp\Big[- z\phi^2 - \frac{a_2}{2}\{(s+\phi)^2 + (s-\phi)^2\}\Big] \\ &= \exp(-a_2 s^2) \int_{-\infty}^{\infty} d\phi \exp[-z\phi^2 - a_2\phi^2] = \exp(-a_2 s^2) I(0). \end{aligned}$$

Therefore $Q'(s)$ reduces to

$$\begin{aligned} Q'(s) &= -q^d \ln\Big[\frac{I(q^{1-d/2}s)}{I(0)}\Big] \\ &= a_2 q^d q^{2-d} s^2 \;=\; a_2' s^2. \end{aligned}$$

Thus the RG transformation for the Gaussian model is

$$\begin{aligned} a_2' &= a_2 q^2 \\ c' &= c. \end{aligned}$$

This is a linear transformation with the fixed point $a_2^* = 0$ and c^* is arbitrary. Hence $\Delta a_2 = a_2 - a_2^*$ has the same transformation law. Therefore the correlation length exponent of the Gaussian model is $\nu = y_1^{-1} = 1/2$. Thus the assumptions made in the derivation of the recursion formulae yield exact results, $\nu = 1/2$ and $\eta = 0$, for the Gaussian model.

5.4 Perturbed Gaussian Model

Now consider the terms with $m = 1$ and $m = 2$ in $Q(s)$, that is

$$Q(s) = a_2 s^2 + a_4 s^4.$$

The quartic term $a_4 s^4$ is taken as a small perturbation. Then one gets

$$\begin{aligned} I(s) = \int_{-\infty}^{\infty} d\phi \exp\Big[- z\phi^2 \quad &- \quad \frac{1}{2}\{a_2(s+\phi)^2 + a_2(s-\phi)^2 \\ &+ \quad a_4(s+\phi)^4 + a_4(s-\phi)^4\}\Big]. \end{aligned}$$

Using the results

$$\begin{aligned} \frac{1}{2}\Big[(s+\phi)^2 + (s-\phi)^2\Big] &= s^2 + \phi^2, \\ \frac{1}{2}\Big[(s+\phi)^4 + (s-\phi)^4\Big] &= s^4 + 6s^2\phi^2 + \phi^4, \end{aligned}$$

$I(s)$ can be simplified as

$$\begin{aligned} I(s) &= \exp(-a_2 s^2 - a_4 s^4) \int_{-\infty}^{\infty} d\phi \exp\Big[- z\phi^2 - \{a_2\phi^2 + 6a_4 s^2\phi^2 + a_4\phi^4\}\Big] \\ &= \exp(-a_2 s^2 - a_4 s^4) \int_{-\infty}^{\infty} d\phi \exp(-\phi^2/\alpha) \\ &\qquad \times \Big[1 - a_4(6s^2\phi^2 + \phi^4) + \frac{1}{2}a_4^2(6s^2\phi^2 + \phi^4)^2 + O(a_4^3)\Big], \end{aligned}$$

where $\alpha = (z + a_2)^{-1}$. The terms containing a_4 have been expanded in Taylor series accurate up to a_4^2. The above expression can be further simplified as

$$\begin{aligned} I(s) &= \exp(-a_2 s^2 - a_4 s^4) \int_{-\infty}^{\infty} d\phi \exp(-\phi^2/\alpha) \\ &\qquad \times \Big[1 - 6a_4 s^2\phi^2 - (a_4 - 18a_4^2 s^4)\phi^4 + 6a_4^2 s^2\phi^6 + \frac{a_4^2}{2}\phi^8\Big]. \end{aligned}$$

If $J_n(\alpha)$ is defined as

$$J_n(\alpha) = \int_{-\infty}^{\infty} d\phi \exp(-\phi^2/\alpha)\phi^n,$$

then one gets

$$\begin{aligned} J_0(\alpha) &= \sqrt{\pi\alpha},\ J_2(\alpha) = \sqrt{\pi\alpha}\frac{\alpha}{2},\ J_4(\alpha) = \sqrt{\pi\alpha}\frac{3}{4}\alpha^2, \\ J_6(\alpha) &= \sqrt{\pi\alpha}\frac{15}{8}\alpha^3,\ J_8(\alpha) = \sqrt{\pi\alpha}\frac{105}{16}\alpha^4. \end{aligned}$$

Therefore $I(s)$ can be evaluated as

$$\begin{aligned} I(s) &= \exp(-a_2 s^2 - a_4 s^4)\sqrt{\pi\alpha} \\ &\times \Big[1 - \frac{6}{2}a_4 s^2\alpha - (a_4 - 18a_4^2 s^4)\frac{3}{4}\alpha^2 + 6a_4^2 s^2 \frac{15}{8}\alpha^3 + \frac{a_4^2}{2}\frac{105}{16}\alpha^4\Big] \\ &= \exp(-a_2 s^2 - a_4 s^4)\sqrt{\pi\alpha} \\ &\times \Big[1 - a_4(\frac{3}{4}\alpha^2 + 3s^2\alpha) + a_4^2(\frac{105}{32}\alpha^4 + \frac{45}{4}\alpha^3 s^2 + \frac{27}{2}\alpha^2 s^4)\Big]. \end{aligned}$$

On taking logarithms,

$$\begin{aligned} -\ln\Big[\frac{I(s)}{I(0)}\Big] &= a_2 s^2 + a_4 s^4 \\ &- \ln\Big[1 - a_4(\frac{3}{4}\alpha^2 + 3s^2\alpha) + a_4^2(\frac{105}{32}\alpha^4 + \frac{45}{4}\alpha^3 s^2 + \frac{27}{2}\alpha^2 s^4)\Big] \\ &+ \ln\Big[1 - a_4\frac{3}{4}\alpha^2 + a_4^2\frac{105}{32}\alpha^4\Big]. \end{aligned}$$

Note that $Q'(s)$ is proportional to $-\ln[I(s)/I(0)]$ and it contains all powers of s^2 even though there were only a quadratic term and a quartic term in $Q(s)$. Thus a large parameter space is to be considered for implementing RG. On expanding the logarithmic terms one finds

$$\begin{aligned} -\ln\Big[\frac{I(s)}{I(0)}\Big] &= a_2 s^2 + a_4 s^4 - \Big[-a_4(\frac{3}{4}\alpha^2 + 3s^2\alpha) \\ &+ a_4^2(\frac{105}{32}\alpha^4 + \frac{45}{4}\alpha^3 s^2 + \frac{27}{2}\alpha^2 s^4) - \frac{a_4^2}{2}(\frac{3}{4}\alpha^2 + 3s^2\alpha)^2\Big] \\ &+ \Big[-a_4\frac{3}{4}\alpha^2 + a_4^2\frac{105}{32}\alpha^4\Big] - \frac{a_4^2}{2}(\frac{3}{4}\alpha^2)^2 \\ &= a_2 s^2 + a_4 s^4 + a_4 3s^2\alpha - a_4^2\Big[\frac{45}{4}\alpha^3 s^2 + \frac{27}{2}\alpha^2 s^4\Big] \\ &+ \frac{a_4^2}{2}\Big[\frac{18}{2}\alpha^3 s^2 + \frac{36}{4}\alpha^2 s^4\Big] + O(a_4^3). \end{aligned}$$

If terms accurate to a_4^2 alone are retained, $Q'(s)$ and $Q(s)$ are of the same form. On introducing the replacement

$$s^2 \to q^{2-d}s^2,$$

one gets

$$\begin{aligned} Q'(s) &= q^d\Big[(a_2 + 3\alpha a_4 - 9a_4^2\alpha^3)q^{2-d}s^2 \\ &+ (a_4 - 9a_4^2\alpha^2)q^{4-2d}s^4\Big] + O(a_4^3) \\ &= a_2' s^2 + a_4' s^4 + O(a_4^3). \end{aligned}$$

Thus the transformation $\mathbf{R}_q$ in the μ-space $\{\mu = (c, a_2, a_4)\}$ is given by

$$\begin{aligned} c' &= c, \\ a_2' &= q^2(a_2 + 3\alpha a_4 - 9a_4^2\alpha^3) + O(a_4^3), \\ a_4' &= q^{4-d}(a_4 - 9a_4^2\alpha^2) + O(a_4^3). \end{aligned}$$

These recursion relations have been derived by assuming that a_4 is a small number and terms of $O(a_4^3)$ and higher orders can be neglected. Therefore the fixed point values a_2^* and a_4^* obtained from them should also be consistent with this approximation. The fixed point value c^* is arbitrary. Other fixed points are defined as

$$\begin{aligned} a_2^* &= q^2\Big[a_2^* + \frac{3a_4^*}{z + a_2^*} - \frac{9{a_4^*}^2}{(z + a_2^*)^3}\Big] + O({a_4^*}^3), \\ a_4^* &= q^{4-d}\Big[a_4^* - \frac{9{a_4^*}^2}{(z + a_2^*)^2}\Big] + O({a_4^*}^3). \end{aligned}$$

Clearly, $a_2^* = 0$ and $a_4^* = 0$ are solutions of these equations. Recall that this solution corresponds to the fixed point of the Gaussian model. From the first equation it is found that

$$(1 - q^2)a_2^* \approx a_4^* + O({a_4^*}^2).$$

Since a_4^* has to be small (due to the nature of the approximate calculation), one should expect a_2^* also to be small. As the aim is to study the recursion formulae near the fixed points, they can be simplified by assuming a_2 and a_4 to be small. Assuming that $a_2 \sim a_4$ and keeping terms up to $O(a_4^2)$ one gets

$$a_2' = q^2\Big[a_2 + \frac{3}{z}a_4(1 - \frac{a_2}{z}) - \frac{9}{z^3}a_4^2\Big] + O(a_4^3), \tag{5.3}$$

$$a_4' = q^{4-d}\Big[a_4 - \frac{9}{z^2}a_4^2 + O(a_4^3\Big]. \tag{5.4}$$

Fixed points of the simplified formulae are given by

$$\begin{aligned} a_2^* &= q^2\Big[a_2^* + \frac{3}{z}a_4^*(1 - \frac{a_2^*}{z}) - \frac{9}{z^3}{a_4^*}^2\Big] + O({a_4^*}^3), \\ a_4^* &= q^{4-d}\Big[a_4^* - \frac{9}{z^2}{a_4^*}^2\Big] + O({a_4^*}^3). \end{aligned}$$

Thus $\mu^* = (c^*, a_2^*, a_4^*) = (c^*, 0, 0)$ is one of the fixed points. This is usually known as the Gaussian fixed point since the probability distribution of the Fourier amplitudes is Gaussian. The second equation shows that the other fixed point value of a_4^* satisfies

$$1 = q^{4-d}\Big[1 - \frac{9}{z^2}a_4^*\Big] + O({a_4^*}^2).$$

That is

$$a_4^* = \frac{z^2}{9}(1 - q^{d-4}) + O({a_4^*}^2).$$

5.5 Dimensionality Expansion

The value of a_4^* must be small for it to be consistent with the approximation scheme, and the only parameter that is free to be adjusted is the spatial dimension d. Note that the parameter $q \approx 2$ as required in the derivation of the recursion formulae. So d is taken as a continuous parameter even though the positive integer values only have physical meaning. The main aim in these considerations is to see what the RG transformation equations can provide in a consistent approximation scheme. Thus the parameter $\epsilon = 4 - d$ is defined as a continuous variable. A small positive value of ϵ implies that $d < 4$. A value $\epsilon = 0.1$ may not have any physical meaning, but keeping d as a parameter of the L-G model, one can investigate the results of a consistent approximation scheme. Assuming that ϵ is small and hence

$$q^{-\epsilon} \approx 1 - \epsilon \ln(q) + O(\epsilon^2),$$

the expression for a_4^* yields

$$a_4^* = \frac{\epsilon}{9} \ln(q) z^2 + O(\epsilon^2).$$

Thus for small ϵ, a_4^* is of order ϵ and hence the approximation scheme is consistent. This observation is the starting point for considering continuous dimension d and "ϵ *expansion*" in the RG approach. For a_2^* one gets

$$\begin{aligned} a_2^* &= \frac{q^2}{1-q^2}\frac{3}{z}a_4^* + O({a_4^*}^2) \\ &= \frac{q^2}{1-q^2}\frac{z}{3}\epsilon \ln(q) + O(\epsilon^2). \end{aligned}$$

Thus another fixed point (depending on the value of ϵ) with a small positive a_4^* and a small negative a_2^* has been obtained. This fixed point will be refered to as the non-Gaussian fixed point. Now, the linearised transformation equations can be investigated around these fixed points.

5.6 Gaussian Fixed Point $\mu^* = (c^*, 0, 0)$

Around this fixed point, it is easy to linearise Eqs.(5.4) and (5.5) to obtain

$$\begin{aligned} \Delta a_2' &= q^2\Big[\Delta a_2 + \frac{3}{z}\Delta a_4\Big], \\ \Delta a_4' &= q^\epsilon \Delta a_4, \ z = ck_m^2\Omega, \quad \epsilon = 4 - d. \end{aligned}$$

Note that the volume $\Omega = (2\pi)^d/V_k$ as well as k_m^2 depend on the parameter q. Assuming that V_k can be approximated as a spherical shell, one gets

$$\frac{1}{z} = \frac{\kappa_d}{ck_m^2} \int_{\Lambda/q}^{\Lambda} k^{d-1} dk,$$

where κ_d is $(2\pi)^{-d}$ times the angular part of the volume in k-space. Since k_m^2 is the mean value of k^2 in Λ/q to Λ, z^{-1} may be written as

$$\begin{aligned} \frac{1}{z} &\approx \frac{\kappa_d}{c} \int_{\Lambda/q}^{\Lambda} \frac{k^{d-1}}{k^2} dk \\ &= \frac{\kappa_d}{c} \frac{1}{d-2} \Big[\Lambda^{d-2} - (\frac{\Lambda}{q})^{d-2} \Big] \\ &= \frac{\kappa_d}{c} \frac{\Lambda^{d-2}}{d-2} (1 - q^{2-d}). \end{aligned}$$

Therefore the transformation equations become

$$\begin{aligned} \Delta a_2' &= q^2 \Delta a_2 + B(q^2 - q^\epsilon) \Delta a_4, \\ \Delta a_4' &= q^\epsilon \Delta a_4, \; B = \frac{3\kappa_d}{(2\pi)^d c} \frac{\Lambda^{d-2}}{d-2}. \end{aligned}$$

The parameter B is independent of q. Thus the linearised RG matrix is

$$\widetilde{\mathbf{R}}_q = \begin{bmatrix} q^2 & B(q^2 - q^\epsilon) \\ 0 & q^\epsilon \end{bmatrix}$$

Its eigenvalues and eigenvectors are

$$\rho_1 = q^{y_1} = q^2, \; \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\rho_2 = q^{y_2} = q^\epsilon, \; \mathbf{e}_2 = \begin{pmatrix} -B \\ 1 \end{pmatrix},$$

Thus both the eigenvalues are greater than unity when $\epsilon > 0$, i.e. $d < 4$. For $d > 4$, one finds that $\rho_1 > 1$ and $\rho_2 < 1$ since $y_1 = 2 > 0$ and $y_2 = \epsilon < 0$. Therefore, the assumptions made in the RG approach regarding the nature of eigenvalues are found to be satisfied for $d > 4$. Thus the fixed point $\mu^* = (c^*, 0, 0)$ is appropriate for $d > 4$ and the correlation length exponent $\nu = y_1^{-1} = 1/2$. The second fixed point is not appropriate for $d > 4$ since a_4^* is negative, a_4^* is proportional to ϵ and $\epsilon < 0$ for $d > 4$, and therefore the fixed point hamiltonian is not normalizable. Any arbitrary deviation

$$\Delta \mathbf{a} = \begin{pmatrix} \Delta a_2 \\ \Delta a_4 \end{pmatrix}$$

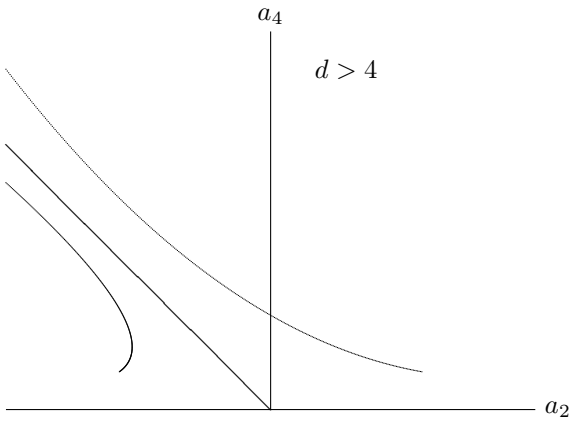

Figure 5.3: Critical Surface.

can be expanded in terms of the eigenvectors as

$$\Delta\mathbf{a} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2.$$

Therefore, the coefficients t_1 and t_2 can be expressed as

$$\begin{aligned} t_2 &= \Delta a_4 \\ t_1 &= \Delta a_2 + B\Delta a_4. \end{aligned}$$

Now, recall that the critical surface is defined as the set of points $\{\Delta\mathbf{a}\}$ which have no projection on the eigenvector $\mathbf{e}_1$. Thus the critical surface is determined by the condition

$$\Delta a_2 + B\Delta a_4 = 0,$$

which is a straight line (see Figure 5.3) in the parameter space. Repeated application (l times) of $\widetilde{\mathbf{R}}_q$ on $\Delta\mathbf{a}$ yields

$$\begin{aligned} \Delta\mathbf{a}^{(l)} &= t_1 q^{2l}\mathbf{e}_1 + t_2 q^{l\epsilon}\mathbf{e}_2 \\ &= (\Delta a_2 + B\Delta a_4)q^{2l}\mathbf{e}_1 + \Delta a_4 q^{l\epsilon}\mathbf{e}_2. \end{aligned}$$

For a large value of l one gets

$$\begin{aligned} \Delta a_2^{(l)} &= (\Delta a_2 + B\Delta a_4)q^{2l} - B\Delta a_4 q^{l\epsilon} \\ &\approx (\Delta a_2 + B\Delta a_4)q^{2l}, \\ \Delta a_4^{(l)} &= \Delta a_4 q^{l\epsilon}, \end{aligned}$$

since $\epsilon < 0$ when $d > 4$. Thus for large l, $\Delta a_4^{(l)}$ tends to zero while $\Delta a_2^{(l)}$ approaches $\pm\infty$ depending on whether

$$t_1 = (\Delta a_2 + B\Delta a_4),$$

is greater or less than zero. The flow of points in the parameter space, on repeated application of $\widetilde{\mathbf{R}}_q$, is also shown in the above figure. Note that points not lying on the critical surface first come closer to the fixed point and then move away from it.

5.7 Failure of Hyperscaling Law for $d > 4$

From the exponents $\nu = 1/2$ and $\eta = 0$, the other exponents can be calculated. They are given by

$$\begin{array}{lll} \alpha = & 2 - \nu d & = 2 - d/2 \\ \gamma = & \nu(2-\eta) & = 1 \\ \beta = & (2-\alpha-\gamma)/2 & = (d-2)/4 \\ \delta = & (\beta+\gamma)/\beta & = (d+2)/(d-2). \end{array}$$

However, several arguments were given earlier (in Chapter 2) to show that Landau's theory together with the Gaussian approximation is correct for $d > 4$ and the exponents are $\nu = 1/2,\ \eta = 0,\ \alpha = 0,\ \gamma = 1,\ \beta = 1/2$ and $\delta = 3$. Thus the values of $\alpha,\ \beta$ and δ disagree with those obtained using the scaling laws. To resolve this point, it is necessary to reexamine the derivation of the hyperscaling law and hence the value of α. The relation connecting the singular parts of free energy densities of two equivalent models generated by RG is

$$F_s(\mu) = q^{-d} F_s(\mu'),$$

where the contribution from the eliminated modes has been omitted. Repeated application of this relation (l times) yields

$$\begin{aligned} F_s(\Delta a_2, \Delta a_4) &= q^{-ld} F_s(\Delta a_2^{(l)}, \Delta a_4^{(l)}) \\ &= q^{-ld} F_s(t_1 q^{2l}, \Delta a_4 q^{l\epsilon}),\ l \gg 1. \end{aligned}$$

Since the initial point is not on the critical surface, $t_1 \neq 0$. With $t_1 \approx A_1 \Delta T$ and small $\Delta T = T - T_c$, one finds

$$F_s(\Delta a_2, \Delta a_4) = q^{-ld} F_s(A_1 \Delta T q^{2l}, \Delta a_4 q^{l\epsilon}).$$

As l increases, $\Delta a_4 q^{l\epsilon}$ tends to zero since $\epsilon < 0$. Then, the choice $q^l = |\Delta T|^{-1/2}$ yields

$$\begin{aligned} F_s(\Delta T) &= |\Delta T|^{d/2} F_s(\pm A_1, 0) \\ &= \textit{constant} \times |\Delta T|^{2-\alpha}. \end{aligned}$$

Thus $2 - \alpha = d/2$ which is a special case of the hyperscaling law with $\nu = 1/2$. However, this conclusion is based on the assumption that $F_s(\pm A_1, 0)$ is a finite number. This requires that the free energy density should remain finite

as the coefficient of the quartic term approaches zero. To check the validity of this assumption, for small $\Delta a_4 q^{l\epsilon}$, the expression for free energy density from Landau's theory can be used. Note that as $d > 4$ and l is a large number, $a_2^{(l)}$ is a large number while $a_4^{(l)}$ is a negligibly small number. Thus for $d > 4$ and large l, the renormalised hamiltonian is a L-G model with a small quartic term. Therefore the idea of linearization around the most probable order parameter may be employed. Thus the use of Landau's expression to obtain the dependence of free energy density on the coefficient of the quartic term may be justified. But Landau's expression for free energy density $F_L(a_2, a_4)$ diverges as a_4^{-1} as $a_4 \to 0$. Thus the assumption that $F_s(\pm A_1, 0)$ is finite and hence the hyperscaling law $2 - \alpha = d/2$ are incorrect. The hyperscaling law is to be rederived by using the fact that $F_L(a_2, a_4)$ diverges as a_4^{-1} as $a_4 \to 0$. Therefore F_s may be written as

$$F_s(A_1 \Delta T q^{2l}, \Delta a_4 q^{l\epsilon}) \approx \frac{1}{\Delta a_4} q^{-l\ \epsilon} F_s^*(A_1 \Delta T q^{2l}, \Delta a_4 q^{l\epsilon}),$$

where F_s^* is finite as l becomes large. Then the functional equation should be written as

$$\begin{aligned} F_s(\Delta a_2, \Delta a_4) &= \frac{1}{\Delta a_4} q^{-ld} q^{-l\ \epsilon} F_s^*(A_1 \Delta T q^{2l}, \Delta a_4 q^{l\epsilon}) \\ &= \frac{1}{\Delta a_4} q^{-4l} F_s^*(A_1 \Delta T q^{2l}, \Delta a_4 q^{l\epsilon}). \end{aligned}$$

Now, choosing $q^l = |\Delta T|^{-1/2}$, for large values of l one gets

$$F_s(\Delta T) = \frac{|\Delta T|^2}{\Delta a_4} F_s^*(\pm A_1, 0).$$

This relation shows that $\alpha = 0$ which further yields $\beta = 1/2$ and $\delta = 3$. Thus the hyperscaling law, derived by assuming that $F_s(\pm A_1, 0)$ is finite, is wrong for $d > 4$ and exponents of Landau's theory are correct.

The variable a_4 is irrelevant for $d > 4$ since it tends to zero on repeated application of the RG transformation. But it is now clear that the free energy density diverges as a_4 approaches zero and this fact makes the hyperscaling law invalid. Therefore a_4 is termed as a dangerous irrelevant variable. Finally, note that the Gaussian fixed point is not appropriate for $d < 4$ since both the eigenvalues are greater than unity.

5.8 Non-Gaussian Fixed Point $\mu^* = (c^*, a_2^*, a_4^*)$

The linearised RG matrix, $\widetilde{\mathbf{R}}_q$, at the non-Gaussian fixed point is to be studied for $d < 4$. The recursion relations

$$a_2' = q^2\Big[a_2 + \frac{3}{z} a_4 (1 - \frac{a_2^*}{z}) - \frac{9}{z^3} a_4^2\Big] + O(a_4^3),$$

$$a_4' = q^{4-d}\Big[a_4 - \frac{9}{z^2}a_4^2\Big] + O(a_4^3),$$

can be linearised by putting $a_2 = a_2^* + \Delta a_2$ etc. That leads to

$$\begin{aligned} a_2^* + \Delta a_2' &= q^2\Big[a_2^* + \Delta a_2 + \frac{3}{z}(a_4^* + \Delta a_4) \\ &\times \{1 - \frac{1}{z}(a_2^* + \Delta a_2)\} - \frac{9}{z^3}({a_4^*}^2 + 2a_4^*\Delta a_4)\Big], \\ a_4^* + \Delta a_4' &= q^\epsilon\Big[a_4^* + \Delta a_4 - \frac{9}{z^2}({a_4^*}^2 + 2a_4^*\Delta a_4)\Big]. \end{aligned}$$

On using the definition of the fixed point values, one finds

$$\begin{aligned} \Delta a_2' &= q^2\Big[\Delta a_2 - \frac{3a_4^*}{z^2}\Delta a_2 + \frac{3}{z}(1 - \frac{a_2^*}{z})\Delta a_4 - \frac{18}{z^3}a_4^*\Delta a_4\Big], \\ \Delta a_4' &= q^\epsilon\Big[\Delta a_4 - \frac{18}{z^2}a_4^*\Delta a_4\Big]. \end{aligned}$$

Thus the transformation matrix is

$$\widetilde{\mathbf{R}}_q = \begin{bmatrix} q^2(1 - 3a_4^*/z^2) & q^2\{(3/z)(1 - a_2^*/z) - 18a_4^*/z\} \\ 0 & q^\epsilon(1 - 18a_4^*/z^2) \end{bmatrix}.$$

The fixed points satisfy the equations

$$\begin{aligned} a_2^* &= \frac{3q^2}{1-q^2}\frac{a_4^*}{z}, \\ a_4^* &= (1 - q^{-\epsilon})\frac{z^2}{9}. \end{aligned}$$

Therefore one finds

$$\begin{aligned} 1 - \frac{3a_4^*}{z^2} &= 1 - \frac{1}{3}(1 - q^{-\epsilon}) \\ &= 1 - \frac{\epsilon}{3}\ln(q) + O(\epsilon^2) \\ &= q^{-\epsilon/3} + O(\epsilon^2), \\ 1 - \frac{18a_4^*}{z^2} &= 1 - 2(1 - q^{-\epsilon}) \\ &= 1 - 2\epsilon\ln(q) + O(\epsilon^2) \\ &= q^{-2\epsilon} + O(\epsilon^2), \\ \frac{3}{z}(1 - \frac{a_2^*}{z}) - 18\frac{a_4^*}{z^3} &= \frac{1}{z}\Big[3\{1 + \frac{\epsilon}{3}\frac{q^2}{q^2-1}\ln(q)\} - 2\epsilon\ln(q)\Big] + O(\epsilon^2) \\ &= \frac{1}{z}\Big[3\{1 - \frac{\epsilon}{3}\ln(q)\} + \frac{\epsilon\ln(q)}{q^2-1}\Big] = \frac{3}{z}q^{-\epsilon/3} + O(\epsilon). \end{aligned}$$

Thus $\widetilde{\mathbf{R}}_q$ is given by

$$\widetilde{\mathbf{R}}_q = \begin{bmatrix} q^{2-\epsilon/3} & (3/z)q^{2-\epsilon/3} + O(\epsilon) \\ 0 & q^{-\epsilon} \end{bmatrix}.$$

Hence the eigenvalues are $\rho_1 = q^{2-\epsilon/3}$ and $\rho_2 = q^{-\epsilon}$ and hence $y_1 = 2 - \epsilon/3$ and $y_2 = -\epsilon$. Thus the fixed point has the required properties for $\epsilon > 0$, i.e. for $d < 4$. It was noted earlier that this fixed point is inappropriate for $d > 4$. The critical exponents can be now computed using the value of y_1. Thus, for $d < 4$ they are given by

$$\begin{aligned}
\eta &= 0 + O(\epsilon^2), \\
\nu &= (2 - \epsilon/3)^{-1} \\
&= 1/2 + \epsilon/12 + O(\epsilon^2), \\
\alpha &= (4 - d)/2 - \epsilon d/12 \\
&= \epsilon/6 + O(\epsilon^2), \\
\gamma &= 2(1/2 + \epsilon/12) \\
&= 1 + \epsilon/6 + O(\epsilon^2), \\
\beta &= (2 - \epsilon/6 - 1 - \epsilon/6)/2 \\
&= 1/2 - \epsilon/6 + O(\epsilon^2), \\
\delta &= (1/2 - \epsilon/6 + 1 + \epsilon/6)/(1/2 - \epsilon/6) \\
&= 3 + \epsilon + O(\epsilon^2),
\end{aligned}$$

Thus, the calculations using the recursion formulae show that for $d < 4$, but $\epsilon = 4 - d$ a small number, α is of order ϵ, η is of order ϵ^2 and δ, γ and ν are greater than those of Landau's theory. The experimental values (for d=3) indeed show that α and η are very small quantities. The increase in the values of other exponents is also in accordance with experimental facts and results of numerical calculations of 3-D Ising model. It should be noted that the eigenvalue ρ_2 becomes unity when the appropriate fixed point changes from Gaussian to non-Gaussian type at $d = 4$.

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Chapter 6

Perturbation Theory and ϵ Expansion

In this chapter, the results obtained using Wilson's recursion formulae are rederived in a more systematic manner. The recursion formulae were derived using a number of approximations and there was no way to assess their accuracy. Further, with the present approach based on perturbation theory, it is possible to develop a systematic procedure for obtaining more accurate results. Recall that a direct perturbation approach, attempted in Chapter 2, is not useful in the critical region. Together with the RG ides, it provides a systematic calculational scheme. Consider the L-G hamiltonian,

$$H[s] = \int_V \Big[\frac{r}{2}\mathbf{s}^2(\mathbf{x}) + \frac{u}{8}\mathbf{s}^4(\mathbf{x}) + \frac{c}{2}(\nabla\mathbf{s}(\mathbf{x}))^2\Big]d\mathbf{x}.$$

Here, the parameters in H have been redefined as $a_2 = r/2, a_4 = u/4$ and c is replaced with $c/2$. Further, H is used to denote H/T for simplifying the notations. The magnetic field term has been omitted since the transformation law for the field is known. The order parameter $\mathbf{s}$ is assumed to have n components. In terms of Fourier amplitudes $s_{i\mathbf{k}}$ in

$$s_i(\mathbf{x}) = \frac{1}{L^{d/2}}\sum_{k\leq\Lambda}\exp(\imath\mathbf{k}\cdot\mathbf{x})s_{i\mathbf{k}}, 1 \leq i \leq n,$$

the hamiltonian can be expressed as

$$\begin{aligned} H[s] &= \frac{1}{2}\sum_i\sum_{k\leq\Lambda}(r+ck^2)|s_{i\mathbf{k}}|^2 \\ &+ \frac{u}{8L^d}\sum_{ij}\sum_{\{k_m\}\leq\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4). \end{aligned}$$

The delta function shows that the sum of wavevectors, $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4$ is zero. Further, the symbol, $\sum_{\{k_m\}}(\cdots)$ indicates summation over each of the

indices k_1 to k_4. The parameter space is $\mu = (r, u, c)$ and $\mu' = \mathbf{R}_q\mu$ is to be obtained by calculating the new hamiltonian $H'[s]$, defined as

$$\begin{aligned} \exp(-H'[s] - AL^d) &= \int \exp(-H[s]) \prod_{i \ \Lambda/q<k\leq\Lambda} ds_{i\mathbf{k}}, \\ s_{i\mathbf{k}} &\rightarrow s_{i\mathbf{k}'} q^{1-\eta/2} \ , \quad \mathbf{k}' = q\mathbf{k}. \end{aligned}$$

Note that the factor $q^{1-\eta/2}$ was obtained from $\alpha(q)q^{d/2}$ where

$$\alpha(q) = q^a, \ a = \frac{1}{2}(2 - \eta - d).$$

The parameter η is to be adjusted to find a proper fixed point and $H'[s] = 0$ when $\mathbf{s} = 0$. The presence of the quartic term in $H[s]$ causes difficulties in implementing the reduction of degrees of freedom. So $\exp(-H[s])$ is expanded as a power series in u. Then it is possible to evaluate all the relevant integrals since they are of the Gaussian form. However, there are enormous complications due to the fact that the Fourier amplitudes with k values in Λ/q to Λ are only to be integrated out. The Gaussian model, which has been discussed earlier, is obtained when $u = 0$. The perturbation expansion of $\exp(-H[s])$ in powers of u implies that the calculations are valid only for small values of u.

6.1 Perturbation Expansion

The range of k values is split into two groups,

$$\begin{aligned} k_A &= \Lambda/q < k \leq \Lambda, \\ k_B &= 0 \leq k \leq \Lambda/q. \end{aligned}$$

$s_{i\mathbf{k}}$ is denoted as $s''_{i\mathbf{k}}$ if k is in k_A group and as $s'_{i\mathbf{k}}$ if k is in k_B group. That is

$$s_{i\mathbf{k}} = \begin{cases} s''_{i\mathbf{k}} & for\ k\ in \quad k_A \\ s'_{i\mathbf{k}} & for\ k\ in \quad k_B \end{cases}$$

Now, each summation in $H[s]$ can be split as

$$\sum_{i \ k}(\cdots) = \sum_{i \ k}^{A}(\cdots) + \sum_{i \ k}^{B}(\cdots).$$

Then $H[s]$ can be written as

$$H[s] = H_0[s''] + H[s'] + H_1[s', s''].$$

The first term is

$$H_0[s''] = \frac{1}{2}\sum_{i \ k}^{A}(r + ck^2)|s''_{i\mathbf{k}}|^2.$$

The second term, which is similar to the L-G hamiltonian, is

$$\begin{aligned} H[s'] &= \frac{1}{2}\sum_{i}^{B}\sum_{k}(r+ck^2)|s'_{i\mathbf{k}}|^2 \\ &+ \frac{u}{8L^d}\sum_{ij}\sum_{\{k_m\}}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}s'_{j\mathbf{k}_3}s'_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \end{aligned}$$

and $H_1[s',s'']$ contains the remaining part in $H[s]$. Clearly, $H_1[s',s'']$ is proportional to u. Therefore it is treated as a perturbation. When the four summations in the quartic term is split into two groups, there are a total of sixteen terms. In one of them, all k_m $(1 \le m \le 4)$ belong to the k_B group and that term is accounted in $H[s']$. The remaining fifteen terms are retained in $H_1[s',s'']$. Thus, at least on summation in $H_1[s',s'']$ contains $s''_{i\mathbf{k}}$ terms. In fact $H_1[s',s'']$ can be written as

$$H_1[s',s''] = \frac{u}{8L^d}\sum_{ij}\sum_{\{k_m\}} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \tag{6.1}$$

where (i) at least one of the k_m $(1 \le m \le 4)$ belongs to the k_A group and hence the corresponding $s_{i\mathbf{k}}$ is $s''_{i\mathbf{k}}$, (ii) each k_m can be either in the k_A group or in the k_B group and hence each $s_{i\mathbf{k}}$ can be either $s''_{i\mathbf{k}}$ or $s'_{i\mathbf{k}}$. With these specifications, it is clear that $H_1[s',s'']$ contains fifteen terms.

Now, the new hamiltonian can be defined as

$$\begin{aligned} \exp(-H'[s]-AL^d) &= \int \exp(-H[s']-H_0[s'']-H_1[s',s'])\prod_{i}\prod_{k_A} ds''_{i\mathbf{k}} \\ &= \exp(-H[s'])\int \exp(-H_0[s'']-H_1[s',s'])\prod_{i}\prod_{k_A} ds''_{i\mathbf{k}}, \\ s'_{i\mathbf{k}} &\to s_{i\mathbf{k}'}q^{1-\eta/2}, \quad \mathbf{k}'=q\mathbf{k}. \end{aligned}$$

where $H[s']$ has been taken outside the integral since it does not contain the integration variable $s''_{i\mathbf{k}}$. Using the notation

$$\begin{aligned} <\cdots> &= \frac{1}{Z_0}\int \exp(-H_0[s''])(\cdots)\prod_{i}\prod_{k_A} ds''_{i\mathbf{k}}, \\ Z_0 &= \int \exp(-H_0[s''])\prod_{i}\prod_{k_A} ds''_{i\mathbf{k}}, \end{aligned}$$

for the averages, one gets

$$\begin{aligned} \exp(-H'[s]-AL^d) &= \exp(-H[s'])<\exp(-H_1[s',s'])>Z_0, \\ s'_{i\mathbf{k}} &\to s_{i\mathbf{k}'}q^{1-\eta/2}, \ \mathbf{k}'=q\mathbf{k}. \end{aligned}$$

Thus the new hamiltonian is given by

$$\begin{aligned} H'[s] + AL^d &= H[s'] - \ln < \exp(-H_1[s', s'']) > - \ln(Z_0), \\ s'_{i\mathbf{k}} &\rightarrow s_{i\mathbf{k}'} q^{1-\eta/2}, \ \mathbf{k}' = q\mathbf{k}. \end{aligned}$$

Now, it is necessary to absorb all terms independent of $s_{i\mathbf{k}'}$ in A since $H'[s]$ should be zero when $s_{i\mathbf{k}'} = 0$. Hence $H'[s]$ is rewritten as

$$\begin{aligned} H'[s] &= H[s'] - \ln < \exp(-H_1[s', s'']) > \\ &+ \ln < \exp(-H_1[0, s'']) >, \\ s'_{i\mathbf{k}} &\rightarrow s_{i\mathbf{k}'} q^{1-\eta/2}, \ \mathbf{k}' = q\mathbf{k}. \end{aligned} \tag{6.2}$$

Then, the free energy density A is given by

$$AL^d = - \ln < \exp(-H_1[0, s'']) > - \ln(Z_0).$$

The last term, $\ln(Z_0)$, has already been calculated in connection with the Gaussian model. Use of that result yields

$$AL^d = - \ln < \exp(-H_1[0, s''] \ ) > - \frac{n}{2} \sum_k^A \ln \Big(\frac{2\pi}{r + ck^2} \Big).$$

The terms in $H'[s]$ can be now analyzed one by one.

(A). Term $H[s']$

The explicit form of $H[s']$ is

$$\begin{aligned} H[s'] &= \frac{1}{2} \sum_{i \ k}^{B} (r + ck^2) |s'_{i\mathbf{k}}|^2 \\ &+ \frac{u}{8L^d} \sum_{ij} \sum_{\{k_m\}}^{B} s'_{i\mathbf{k}_1} s'_{i\mathbf{k}_2} s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4), \end{aligned}$$

where $s'_{i\mathbf{k}_m}$ is to be replaced with $s_{i\mathbf{k}'_m} q^{1-\eta/2}$, $\mathbf{k}'_m = q\mathbf{k}_m$. If k_m is in k_B group, then k'_m takes values in the full range $0 \leq k \leq \Lambda$. Therefore one finds

$$\begin{aligned} H[s'] &= \frac{1}{2} \sum_{i \ k' \leq \Lambda} (r + cq^{-2} k'^2) q^{2-\eta} |s_{i\mathbf{k}'}|^2 \\ &+ \frac{u \ q^{4-2\eta}}{8(L'q)^d} \sum_{ij} \sum_{\{k'_m\} \leq \Lambda} s_{i\mathbf{k}'_1} s_{i\mathbf{k}'_2} s_{j\mathbf{k}'_3} s_{j\mathbf{k}'_4} \delta(\mathbf{k}'_1 + \mathbf{k}'_2 + \mathbf{k}'_3 + \mathbf{k}'_4), \end{aligned}$$

where the substitutions $\mathbf{k}_m = \mathbf{k}'_m / q$ and $L = qL'$ have been made. Thus

$$H[s'] = \frac{1}{2} \sum_{i \ k \leq \Lambda} (rq^{2-\eta} + cq^{-\eta} k^2) |s_{i\mathbf{k}}|^2 \tag{6.3}$$

$$+ \quad \frac{u\ q^{4-d-2\eta}}{8L'^d} \sum_{ij} \sum_{\{k_m\}\leq\Lambda} s_{i\mathbf{k}_1} s_{i\mathbf{k}_2} s_{j\mathbf{k}_3} s_{j\mathbf{k}_4} \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4),$$

where the dummy variable $\mathbf{k}'_m$ has been replaced by $\mathbf{k}_m$.

(B). Term $-\ln < \exp(-H_1[s',s'']) > + \ln < \exp(-H_1[0,s'']) >$

Assuming u to be small, to second order accuracy in u, one finds

$$\begin{aligned} \ln < \exp(-H_1[s',s'']) > &= \ln \Big[1 - < H_1[s',s''] > \\ &+ \frac{1}{2} < H_1^2[s',s''] > + O(u^3) \Big] \\ &= - < H_1[s',s''] > + \frac{1}{2} < H_1^2[s',s''] > \\ &- \frac{1}{2} < H_1[s',s''] >^2 + O(u^3). \end{aligned}$$

In the last expression, $\ln(1+x)$ is approximated as $x - x^2/2 + O(x^3)$. In a similar way

$$\begin{aligned} \ln < \exp(-H_1[0,s'']) > &= - < H_1[0,s''] > + \frac{1}{2} < H_1^2[0,s''] > \\ &- \frac{1}{2} < H_1[0,s''] >^2 + O(u^3). \end{aligned}$$

The two expressions give

$$\begin{aligned} & -\ln < \exp(-H_1[s',s'']) > + \ln < \exp(-H_1[0,s'']) > \\ &= < H_1[s',s''] > - < H_1[0,s''] > \\ &- \frac{1}{2} \Big(< H_1^2[s',s''] > - < H_1^2[0,s''] > \Big) \\ &+ \frac{1}{2} \Big(< H_1[s',s''] >^2 - < H_1[0,s''] >^2 \Big) + O(u^3). \end{aligned}$$

Now, let $\widetilde{H}_1$ be defined as

$$\widetilde{H}_1[s',s''] = H_1[s',s''] - H_1[0,s'']. \tag{6.4}$$

Then, the previous expression becomes

$$\begin{aligned} & -\ln < \exp(-H_1[s',s'']) > + \ln < \exp(-H_1[0,s'']) > \\ &= < \widetilde{H}_1[s',s''] > - \frac{1}{2} \Big(< \widetilde{H}_1^2[s',s''] > - < \widetilde{H}_1[0,s''] >^2 \Big) \\ &- \Big(< \widetilde{H}_1[s',s''] \widetilde{H}_1[0,s''] > - < \widetilde{H}_1[s',s''] > < \widetilde{H}_1[0,s''] > \Big) + O(u^3). \end{aligned} \tag{6.5}$$

All the four k_m's in $\widetilde{H}_1[s', s'']$ do not belong to the k_A group. In fact, all k_m's in $H_1[0, s'']$ are in the k_A group. Thus $\widetilde{H}_1[s', s'']$ represents fourteen terms out of the fifteen terms in $H_1[s', s'']$. Using Eqs. (6.2), (6.3) and (6.5), the new hamiltonian can be written as

$$\begin{aligned} H'[s] &= \frac{1}{2}\sum_{i}\sum_{k\le\Lambda}(rq^{2-\eta}+cq^{-\eta}k^2)|s_{i\mathbf{k}}|^2 \qquad (6.6)\\ &+ \frac{u\ q^{4-d-2\eta}}{8L'^d}\sum_{ij}\sum_{\{k_m\}\le\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\\ &+ <\widetilde{H}_1[s', s'']> -\frac{1}{2}\Big(<\widetilde{H}_1^2[s', s'']> - <\widetilde{H}_1[s', s'']>^2\Big)\\ &- \Big[<\widetilde{H}_1[s', s'']\widetilde{H}_1[0, s'']> - <\widetilde{H}_1[s', s'']><\widetilde{H}_1[0, s'']>\Big] + O(u^3). \end{aligned}$$

After calculating the averages, the renormalization of the Fourier amplitudes is to be effected with the replacement $s'_{i\mathbf{k}} \to s_{i\mathbf{k}'}q^{1-\eta/2}$. The averages to be calculated are with the Gaussian distribution $\exp(-H_0)$. So the following properties are useful.

6.1.1 Averaging with $\exp(-H_0[s''])$

(i) First of all, note that

$$\begin{aligned} <s''_{i\mathbf{k}_1}> &= \frac{1}{Z_0}\int \exp(-H_0[s''])s''_{i\mathbf{k}_1}\prod_i^A\prod_k ds''_{i\mathbf{k}}\\ &= \frac{1}{Z_{01}}\int \exp\Big[-\frac{1}{2}(r+ck_1^2)|s''_{i\mathbf{k}_1}|^2\Big]s''_{i\mathbf{k}_1}ds''_{i\mathbf{k}_1}, \end{aligned}$$

where Z_{01} is given by

$$Z_{01} = \int \exp\Big[-\frac{1}{2}(r+ck_1^2)|s''_{i\mathbf{k}_1}|^2\Big]ds''_{i\mathbf{k}_1}.$$

On separating $s''_{i\mathbf{k}_1}$ to real and imaginary parts, one easily finds that the average $<s''_{i\mathbf{k}_1}>=0$.

(ii) In a similar way, it can be shown that

$$\begin{aligned} <s''_{i\mathbf{k}_1}s''_{j\mathbf{k}_1}> &= 0\ for\ i\ \neq j,\\ <s''_{i\mathbf{k}_1}s''_{i\mathbf{k}_2}> &= 0\ for\ \mathbf{k}_1+\mathbf{k}_2\ \neq 0. \end{aligned}$$

For $\mathbf{k}_2 = -\mathbf{k}_1$, the amplitudes are complex conjugates and hence

$$\begin{aligned} <s''_{i\mathbf{k}_1}s''_{i\ -\mathbf{k}_1}> &= \frac{1}{Z_0}\int \exp\Big[-\frac{1}{2}(r+ck_1^2)|s''_{i\mathbf{k}_1}|^2\Big]|s''_{i\mathbf{k}_1}|^2ds''_{i\mathbf{k}_1}\\ &= 2\frac{\int x^2\exp[(r+ck_1^2)x^2]dx}{\int \exp[(r+ck_1^2)x^2]dx} \end{aligned}$$

$$\begin{aligned} &= 2\frac{(\sqrt{\pi}/2)(r+ck_1^2)^{-3/2}}{\sqrt{\pi}(r+ck_1^2)^{-1/2}} \\ &= (r+ck_1^2)^{-1}. \end{aligned}$$

Thus, the general result is

$$\begin{aligned} < s''_{i\mathbf{k}_1} s''_{j\mathbf{k}_2} > &= \delta_{ij}\delta(\mathbf{k}_1+\mathbf{k}_2)G_0(k_1), \\ G_0(k_1) &= (r+ck_1^2)^{-1}. \end{aligned}$$

(iii) Another property is that

$$< s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} \cdots s''_{i\mathbf{k}_{2l+1}} >= 0.$$

(iv) When there are an even number of terms in the product,

$$\begin{aligned} &< s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} \cdots s''_{i\mathbf{k}_{2l}} > \\ &= \sum_{all\ \ pairs} < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} >< s''_{i\mathbf{k}_3} s''_{i\mathbf{k}_4} > \cdots < s''_{i\mathbf{k}_{2l-1}} s''_{i\mathbf{k}_{2l}} > . \end{aligned}$$

Thus, for example

$$\begin{aligned} < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} s''_{i\mathbf{k}_3} s''_{i\mathbf{k}_4} > &= < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} >< s''_{i\mathbf{k}_3} s''_{i\mathbf{k}_4} > \\ &+ < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_3} >< s''_{i\mathbf{k}_2} s''_{i\mathbf{k}_4} > \\ &+ < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_4} >< s''_{i\mathbf{k}_2} s''_{i\mathbf{k}_3} > . \end{aligned}$$

With these results, the renormalised hamiltonian in the first order approximation can be calculated easily.

6.2 First Order Approximation to $H'[s]$

At this order of approximation, it is enough to calculate $< \widetilde{H}_1[s', s''] >$. From Eqs. (6.1) and (6.4), one finds

$$\widetilde{H}_1[s', s''] = \frac{u}{8L^d} \sum_{ij} \sum_{\{k_m\}} s_{i\mathbf{k}_1} s_{i\mathbf{k}_2} s_{j\mathbf{k}_3} s_{j\mathbf{k}_4} \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4),$$

where, (i) at least one of the $k_m (1 \le m \le 4)$ is in the k_A group, (ii) at least one of the k_m is in the k_B group, and (iii) each k_m can be either in the k_A group or in the k_B group. Note that if only one k_m is in the k_A group, then $< \widetilde{H}_1 >= 0$. Similarly, if three k_m are in the k_A group, then again $< \widetilde{H}_1 >= 0$. There is no possibility of all four k_m appearing in the k_A group. Thus it is enough to consider the case of two k_m in the k_A group. Out of the four k_m, two in the k_A group can be chosen $4C_2 = 6$ ways. That is, any one of the pairs, $(k_1k_2), (k_1k_3), (k_1k_4), (k_2k_3), (k_2k_4)$ or (k_3k_4) can be in the k_A group. Also note that if the pair (k_1k_2) is in the k_A group, then the pair (k_3k_4) is in the k_B group. Therefore $< \widetilde{H}_1 >$ is found to be

$$
\begin{aligned}
< \widetilde{H}_1 > &= \frac{u}{8L^d} \sum_{ij} \Big[\sum_{k_1 k_2}^{A} < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} > \sum_{\mathbf{k}_3 \mathbf{k}_4}^{B} s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4} \qquad (6.7) \\
&+ \sum_{k_1 k_3}^{A} < s''_{i\mathbf{k}_1} s''_{j\mathbf{k}_3} > \sum_{k_2 k_4}^{B} s'_{i\mathbf{k}_2} s'_{j\mathbf{k}_4} + \sum_{k_1 k_4}^{A} < s''_{i\mathbf{k}_1} s''_{j\mathbf{k}_4} > \sum_{k_2 k_3}^{B} s'_{i\mathbf{k}_2} s'_{j\mathbf{k}_3} \\
&+ \sum_{k_2 k_3}^{A} < s''_{i\mathbf{k}_2} s''_{j\mathbf{k}_3} > \sum_{k_1 k_4}^{B} s'_{i\mathbf{k}_1} s'_{j\mathbf{k}_4} + \sum_{k_2 k_4}^{A} < s''_{i\mathbf{k}_2} s''_{j\mathbf{k}_4} > \sum_{k_1 k_3}^{B} s'_{i\mathbf{k}_1} s'_{j\mathbf{k}_3} \\
&+ \sum_{k_3 k_4}^{A} < s''_{j\mathbf{k}_3} s''_{j\mathbf{k}_4} > \sum_{k_1 k_2}^{B} s'_{i\mathbf{k}_1} s'_{i\mathbf{k}_2} \Big] \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4).
\end{aligned}
$$

Since the variables in the summation are dummy variables, one finds that the first and last terms are the same while the remaining four are similar to each other. That means

$$
\begin{aligned}
< \widetilde{H}_1[s', s''] > &= \frac{u}{8L^d} \sum_{ij} \Big[2 \sum_{k_1 k_2}^{A} < s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} > \sum_{k_3 k_4}^{B} s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4} \\
&+ 4 \sum_{k_1 k_3}^{A} < s''_{i\mathbf{k}_1} s''_{j\mathbf{k}_3} > \sum_{k_2 k_4}^{B} s'_{i\mathbf{k}_2} s'_{j\mathbf{k}_4} \Big] \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4).
\end{aligned}
$$

Substitution of the expressions for the averages leads to

$$
\begin{aligned}
< \widetilde{H}_1 > &= \frac{u}{8L^d} \sum_{ij} \Big[2 \sum_{k_1 k_2}^{A} \delta_{ii} \delta(\mathbf{k}_1 + \mathbf{k}_2) G_0(k_1) \sum_{k_3 k_4}^{B} s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4} \\
&+ 4 \sum_{k_1 k_3}^{A} \delta_{ij} \delta(\mathbf{k}_1 + \mathbf{k}_3) G_0(k_1) \sum_{k_2 k_4}^{B} s'_{i\mathbf{k}_2} s'_{j\mathbf{k}_4} \Big] \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4).
\end{aligned}
$$

On simplifying the sums one gets

$$
\begin{aligned}
< \widetilde{H}_1 > &= \frac{u}{8L^d} \Big[2n \sum_{k_1}^{A} G_0(k_1) \sum_{j} \sum_{k_3}^{B} |s'_{j\mathbf{k}_3}|^2 \\
&+ 4 \sum_{k_1}^{A} G_0(k_1) \sum_{i} \sum_{k_2}^{B} |s'_{i\mathbf{k}_2}|^2 \Big].
\end{aligned}
$$

Since both terms are of the same type, they can be rewritten as

$$
< \widetilde{H}_1 >= \frac{u}{8L^d} (2n + 4) \sum_{k_1}^{A} G_0(k_1) \sum_{ik_2}^{B} |s'_{i\mathbf{k}_2}|^2.
$$

With the replacement $s'_{i\mathbf{k}_2} \rightarrow s_{i\mathbf{k}'_2} q^{1-\eta/2}$, $\mathbf{k}'_2 = q\mathbf{k}_2$ runs over the full k-space and hence

$$
< \widetilde{H}_1[s', s''] >= \frac{u}{8} (2n + 4) \sum_{i} \sum_{k \leq \Lambda} |s_{i\mathbf{k}}|^2 q^{2-\eta} [\frac{1}{L^d} \sum_{k_1}^{A} G_0(k_1)].
$$

Thus, in the first order approximation, Eq.(6.6) for $H'[s]$ becomes

$$\begin{aligned} H'[s] &= \frac{1}{2}\sum_{i}\sum_{k\le\Lambda}(rq^{2-\eta}+cq^{-\eta}k^2)|s_{i\mathbf{k}}|^2 \\ &+ \frac{u}{8L'^d}q^{4-d-2\eta}\sum_{ij}\sum_{\{k_m\}\le\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \\ &+ \frac{u\ q^{2-\eta}}{8}(2n+4)\sum_{i}\sum_{k\le\Lambda}|s_{i\mathbf{k}}|^2[\frac{1}{L^d}\sum_{k_1}^{A}G_0(k_1)] + O(u^2). \end{aligned} \tag{6.8}$$

The above expression looks exactly like the starting L-G hamiltonian. In terms of the parameter set $\mu' = (r', u', c')$, it can be rewritten as

$$\begin{aligned} H'[s] &= \frac{1}{2}\sum_{i}\sum_{k\le\Lambda}(r'+c'k^2)|s_{i\mathbf{k}}|^2 \\ &+ \frac{u'}{8L'^d}\sum_{ij}\sum_{\{k_m\}\le\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) + O(u^2). \end{aligned}$$

Thus the transformation equations of the parameters are

$$\begin{aligned} r' &= rq^{2-\eta} + \frac{u}{4}(2n+4)q^{2-\eta}[L^{-d}\sum_{ik_1}^{A}G_0(k_1)] + O(u^2), \\ u' &= uq^{4-d-2\eta} + O(u^2), \\ c &= cq^{-\eta} + O(u^2). \end{aligned}$$

The thermodynamic limit is obtained with the replacement

$$\frac{1}{L^d}\sum_{k_1}^{A}G_0(k_1) \to \frac{1}{(2\pi)^d}\int^{A}G_0(k_1)d\mathbf{k}_1,$$

where the superscript on the integral denotes that it is over the range of the k_A group. Thus, the transformations reduce to

$$\begin{aligned} r' &= q^{2-\eta}\Big[r + \frac{u}{2}\frac{n+2}{(2\pi)^d}\int^{A}G_0(k_1)d\mathbf{k}_1\Big] + O(u^2), \\ u' &= uq^{4-d-2\eta} + O(u^2), \\ c' &= cq^{-\eta} + O(u^2). \end{aligned}$$

As seen earlier, for these equations (defining $\mathbf{R}_q$) to have a fixed point, η should be chosen as zero and hence $c' = c$. The other two equations show that the fixed point values of r and u are $r^* = u^* = 0$. Noting that

$$G_0(k_1) = \frac{1}{r + ck_1^2},$$

and linearising around $\mu^* = (0, 0, c)$, one gets

$$\begin{aligned}\Delta r' &= q^2\Big[\Delta r + \frac{\Delta u}{2}\frac{n+2}{(2\pi)^d}\int^A \frac{d\mathbf{k}_1}{ck_1^2}\Big] + O(u^2),\\ \Delta u' &= \Delta u q^{4-d} + O(u^2),\end{aligned}$$

where terms like $(\Delta r \Delta u)$ and higher order terms are neglected. The integral can be evaluated as

$$\frac{1}{(2\pi)^d}\int^A \frac{d\mathbf{k}_1}{k_1^2} = \kappa_d \int_{\Lambda/q}^{\Lambda} \frac{k_1^{d-1}}{k_1^2} dk_1 = \kappa_d \frac{\Lambda^{d-2}}{d-2}(1 - q^{2-d}),$$

where $\kappa_d = (2\pi)^{-d} 2\pi^{d/2}\Gamma(d/2)$. The linearised transformations are thus given by

$$\begin{aligned}\Delta r' &= q^2\Big[\Delta r + \frac{\Delta u}{2c}(n+2)\kappa_d \frac{\Lambda^{d-2}}{d-2}(1 - q^{2-d})\Big] + O(u^2),\\ \Delta u' &= \Delta u q^{4-d} + O(u^2).\end{aligned}$$

The eigenvalues of the matrix are $\rho_1 = q^2$ and $\rho_2 = q^{4-d}$. Note that these results are exactly the same as those obtained with Wilson's recursion formulae. However, they have been derived here without the ad hoc assumptions in the derivation of the recursion formulae. Thus one concludes that these results are appropriate for $d > 4$ and they yield Landau's theory exponents.

6.2.1 Effect of $\mathbf{s}^6$ in $H'[s]$

Suppose an additional term

$$v\int_V \mathbf{s}^6(\mathbf{x})d\mathbf{x} = \frac{v}{L^{2d}}\sum_{ijl}\sum_{k_1\cdots k_6\le\Lambda} s_{i\mathbf{k}_1}\cdots s_{l\mathbf{k}_6}\delta(\mathbf{k}_1 + \cdots + \mathbf{k}_6),$$

is added to the L-G hamiltonian. Then, the recursion relations are

$$\begin{aligned}r' &= rq^{2-\eta} + D_1 u + D_2 v + O(u^2) + O(v^2),\\ u' &= uq^{4-d-2\eta} + E_1 v + O(u^2) + O(v^2),\\ v' &= vq^{6-2d-3\eta} + O(u^2) + O(v^2),\\ c' &= cq^{-\eta} + O(u^2) + O(v^2),\end{aligned}$$

where D_1, D_2, and E_1 are dependent on r. They show that η should be chosen to be zero and the fixed point values are $r^* = u^* = v^* = 0$. The eigenvalues of the linearised transformation are given by

$$\rho_1 = q^2,\ \rho_2 = q^{4-d},\ \rho_3 = q^{6-2d}.$$

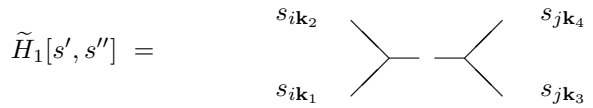

Figure 6.1: Diagram for $\widetilde{H}_1$.

Thus, all the eigenvalues, except ρ_1 are less than unity for $d > 4$. In fact, one can start with a very general hamiltonian containing all even powers of $\mathbf{s}$ and reach the conclusion that the fixed point $\mu^* = (0, 0, \cdots, c)$ is appropriate for $d > 4$. The corresponding fixed point hamiltonian is $\int c(\nabla \mathbf{s})^2 d\mathbf{x}$ and the exponents are the same as those of Landau's theory. The first order approximation is inappropriate for $d \leq 4$. For instance, $\rho_2 = 1$ for $d = 3$. The next higher order approximation is necessary to find a suitable fixed point for $d \leq 4$.

6.2.2 Diagramatic Method

The main result of the first order approximation is the expression for $< \widetilde{H}_1 >$ given in Eq.(6.7). A diagramatic method for obtaining the same result is developed below. A diagram representing $\widetilde{H}_1$ is constructed in the following way. Make a broken line representing the factor u in $\widetilde{H}_1$. At each end of the line, put two solid lines. Thus there are a total of four solid lines. The two solid lines at one end denote $s_{i\mathbf{k}_1}$ and $s_{i\mathbf{k}_2}$. The diagram so obtained for $\widetilde{H}_1$ is shown in Figure 6.1. The delta function in $\widetilde{H}_1$ shows that the sum of the wave vectors in the diagram should be zero. Recall that out of the four s terms, at least one should be s' and another should be s''. Each k_m can be either in the k_A group or in the k_B group. Hence each solid line can represent a s' or a s''. Now, it is possible to choose one s' in $4C_1 = 4$ ways, two s' in $4C_2 = 6$ ways and three s' in $4C_3 = 4$ ways. Thus the total of $4 + 6 + 4 = 14$ ways represent the fourteen terms in $\widetilde{H}_1$. In $< \widetilde{H}_1 >$, there will not be any s''. In fact, they are averaged with the Gaussian distribution $\exp(-H_0[s''])$. The rules for finding the averages with the Gaussian distribution can be translated to the diagramatic language in the following way.

Rule 1 Any solid line representing a s'' should be connected to another solid line representing a different s''.

If there is only one s'', then according to this rule, it can not be connected. Thus there is no contribution to $< \widetilde{H}_1 >$ if there is only one s''. Similarly, in the case when there are three s'', there is no contribution to the average since

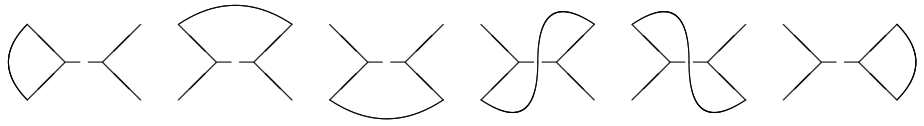

Figure 6.2: Diagrams in $< \widetilde{H}_1[s', s''] >$.

one of the s'' always remains unconnected. If there are two s'', there are six ways of choosing them and by using rule-1, one obtains the six diagrams shown in Figure 6.2. Each diagram gives a contribution to $< \widetilde{H}_1 >$. The diagrams, in the order, represent the six terms in $< \widetilde{H}_1 >$ given explicitly in Eq.(6.7). In the first and last diagrams, there are closed loops, i.e. starting from any point on the loop, one can come back to the same point without going over the broken line representing u. The corresponding terms in Eq.(6.7) have a factor n. This observation leads to the second rule.

Rule 2 The closed loop contributes a weight n (number of order parameter components) and the open loop contributes a weight unity to the average.

On applying this rule one finds that there is a factor $(2n + 4)$ in $< \widetilde{H}_1 >$. The remaining factors in it can be obtained by explicitly writing down the contribution from any of the diagrams using rule-3.

Rule 3 If two solid lines representing two s'' (say, $s''_{i\mathbf{k}_1}$ and $s''_{j\mathbf{k}_2}$) are connected , then, replace $s''_{i\mathbf{k}_1} s''_{j\mathbf{k}_2}$ in $\widetilde{H}_1$ by $\delta_{ij} G_0(k_1)\delta(\mathbf{k}_1 + \mathbf{k}_2)$.

Applying this rule to the first diagram in Figure 6.2, one finds that

$$\begin{aligned}
&\sum_{ij}\sum_{k_1k_2}^{A}\sum_{k_3k_4}^{B} s''_{i\mathbf{k}_1} s''_{i\mathbf{k}_2} s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \\
&\quad\rightarrow \quad \sum_{ij}\sum_{k_1k_2}^{A}\sum_{k_3k_4}^{B} \delta_{ij} G_0(k_1)\delta(\mathbf{k}_1+\mathbf{k}_2) s'_{j\mathbf{k}_3} s'_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \\
&\quad= \quad n\sum_{j}\sum_{k_3}^{B} |s'_{j\mathbf{k}_3}|^2 \sum_{k_1}^{A} G_0(k_1).
\end{aligned}$$

Now, replacing the factor n, which has come because the first diagram has a closed loop, with $(2n + 4)$ and putting the constant factor $(u/8)L^{-d}$ leads to

$$< \widetilde{H}_1[s', s''] >= \frac{u}{8L^d}(2n+4)\sum_{jk_3}^{B} |s'_{j\mathbf{k}_3}|^2 \sum_{k_1}^{A} G_0(k_1),$$

which is same as the result obtained earlier. In the first order approximation, there are only six terms in $< \widetilde{H}_1 >$ and every step can be written down

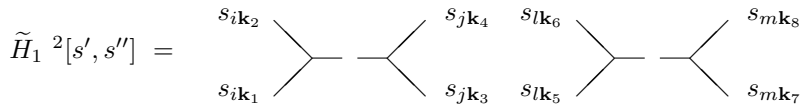

Figure 6.3: Diagram for $\widetilde{H}_1^2$.

explicitly. But, for calculating the second order terms, the rules formulated above turn out to be essential.

6.3 Second Order Approximation to H'[s]

Now, terms in the last two square brackets in Eq.(6.6) are to be calculated. First of all, consider the first square bracket.

(A). $\frac{1}{2}\Big[< \widetilde{H}_1^2[s',s''] > - < \widetilde{H}_1[s',s''] >^2 \Big]$

The term $< \widetilde{H}_1 >$ is already calculated. So one has

$$\frac{1}{2} < \widetilde{H}_1[s',s''] >^2 = \frac{1}{2}(\frac{u}{8})^2(2n+4)^2\Big[\sum_{ik_1}^{B}|s'_{j\mathbf{k}_1}|^2\Big]^2\Big[L^{-d}\sum_{k_2}^{A}G_0(k_2)\Big]^2,$$

For $\frac{1}{2}\widetilde{H}_1^2$, the expression is

$$\begin{aligned}\frac{1}{2}\widetilde{H}_1^2[s',s''] &= \frac{1}{2}(\frac{u}{8})^2L^{-2d}\sum_{ij}\sum_{\{k_x\}\le\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\\ &\times \sum_{lm}\sum_{\{k_y\}\le\Lambda} s_{l\mathbf{k}_5}s_{l\mathbf{k}_6}s_{m\mathbf{k}_7}s_{m\mathbf{k}_8}\delta(\mathbf{k}_5+\mathbf{k}_6+\mathbf{k}_7+\mathbf{k}_8).\end{aligned}$$

The diagram for $\widetilde{H}_1^2$ is shown in Figure 6.3 where each broken line represents a factor u. Each solid line can represent a s' or s''. There should be at least one s' and one s'' in each part of the diagram. The sum of the wave vectors of each part should be zero. To calculate the average, the wave vector sets $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ and $(\mathbf{k}_5,\mathbf{k}_6,\mathbf{k}_7,\mathbf{k}_8)$ are denoted as group-1 and group-2 respectively. Table 6.1 shows the various ways of choosing s'' terms in group-1 and group-2.

From there, it is clear that the cases to be considered are $(N_1,N_2) = (1,1)$, $(2,2)$ and $(3,3)$ where N_1 and N_2 are the number of k's (with values in the k_A range) belonging to group-1 and group-2 respectively. There are

Table 6.1: Various Choices of s''.

N_1=Number of k in the k_A range belonging to group-1.
N_2=Number of k in the k_A range belonging to group-2.
N_3=Total number of k in the k_B range.

N_1	N_2	N_3	*Remarks*
1	1	6	6 s' *after averaging. Hence a term* $\mathbf{s}^6$ *in* H'.
1	2	5	*Odd no. of* k *in* k_A *group. No contribution.*
1	3	4	*No contribution (see discussion below).*
2	1	5	*Odd no. of* k *in* k_A *group. No contribution.*
2	2	4	4 s' *after averaging. Hence a term* $\mathbf{s}^6$ *in* H'.
2	3	3	*Odd no. of* k *in* k_A *group. No contribution.*
3	1	4	*No contribution (see discussion below).*
3	2	3	*Odd no. of* k *in* k_A *group. No contribution.*
3	3	2	2 s' *after averaging. Hence a term* $\mathbf{s}^2$ *in* H'.

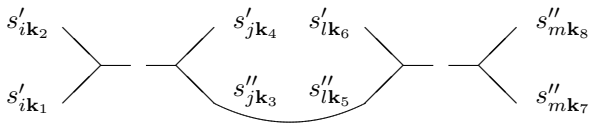

Figure 6.4: One way of connecting two s'' in group-2.

no contributions to the average from the cases $(N_1, N_2) = (1, 3)$ and $(3, 1)$ due to the following reason. When $(N_1, N_2) = (1, 3)$, there will be one s'' in group-1 and three s'' in group-2. Every s'' should be connected to another s'' to obtain a non-zero contribution. Therefore, in group-2 itself, two s'' are to be connected. A possible way of making this connection, shown in Figure 6.4, leads to $\mathbf{k}_7 + \mathbf{k}_8 = 0$. This further leads to $\mathbf{k}_5 + \mathbf{k}_6 = 0$ so that the total sum is zero. But, this condition can not be satisfied since $\mathbf{k}_5$ is in the k_A group and $\mathbf{k}_6$ is in the k_B group. Thus, the case $(N_1, N_2) = (1, 3)$ and, in a similar manner, $(N_1, N_2) = (3, 1)$ do not contribute to the average. The case $(N_1, N_2) = (1, 1)$ yields six s' terms after averaging and hence leads to a term $\int \mathbf{s}^6 d\mathbf{x}$ in the new hamiltonian. This is the way new types of terms are generated by the RG transformation. This term is not considered since the transformation of interest is that of the parameter set $\mu = (r, u, c)$ in the L-G hamiltonian. Thus it is sufficient to consider the cases $(N_1, N_2) = (2, 2)$ and $(3, 3)$.

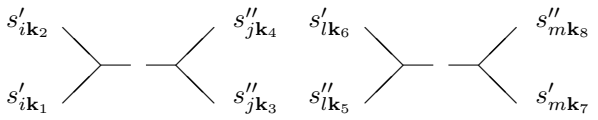

Figure 6.5: A typical Diagram in $\widetilde{H}_1^2[s', s'']$.

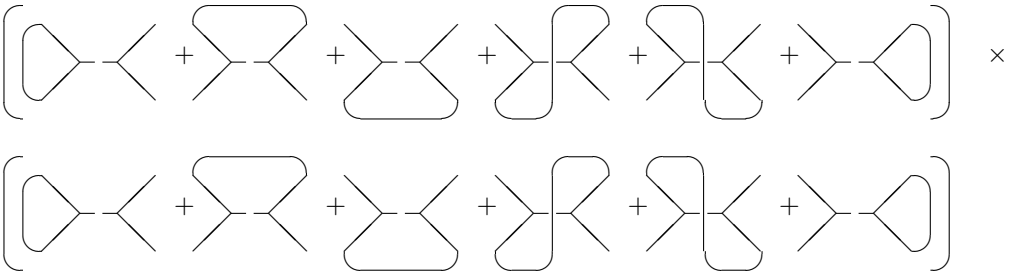

Figure 6.6: A set of diagram in $< \widetilde{H}_1^2[s', s''] >$.

Case $(N_1, N_2) = (2, 2)$

In this case, there are two s'' in group-1 and group-2. A typical diagram for $\widetilde{H}_1^2$ is shown in Figure 6.5. First of all, the s'' terms in group-1 and group-2 can be connected among themselves. Various connected diagrams obtained in this way are shown in Figure 6.6. As seen from the calculation of $< \widetilde{H}_1 >$, each diagram contributes a factor

$$t = \frac{u}{8L^d} \sum_{jk_3}^{B} |s'_{j\mathbf{k}_3}|^2 \sum_{k_1}^{A} G_0(k_1).$$

Using the rule that a closed loop has a weight n and an open loop has a weight unity, the total weight is found to be

$$n(2n+4) + 1(2n+4) + \cdots n(2n+4) = (2n+4)^2.$$

Thus the above way of pairing yields

$$t_1 \;=\; (\frac{u}{8})^2 L^{-2d} (2n+4)^2 \Big[\sum_{jk_3}^{B} |s'_{j\mathbf{k}_3}|^2 \Big]^2 \Big[\sum_{k_1}^{A} G_0(k_1) \Big]^2.$$

Note that this contribution is just $< \widetilde{H}_1 >^2$. Thus pairing of s'' terms among themselves yields a contribution which cancels with $- < \widetilde{H}_1 >^2$.

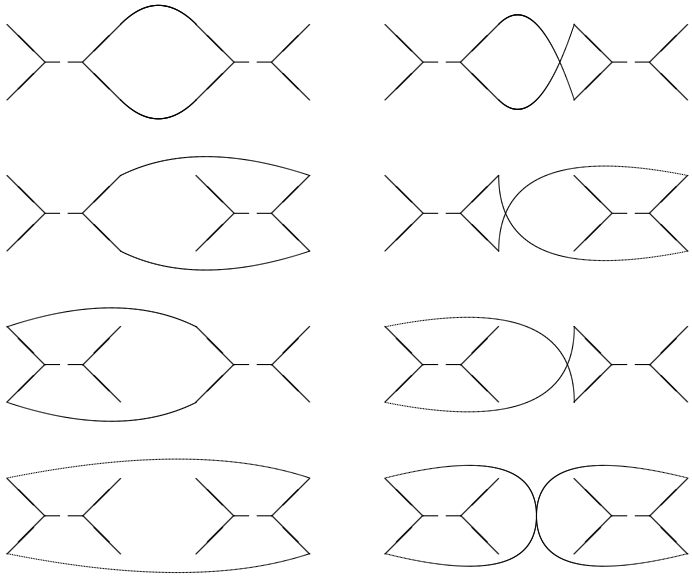

Figure 6.7: Diagrams in $< \widetilde{H}_1[s', s'']^2 >$.

Now, consider the ways of connecting a s'' from group-1 with another one in group-2. Two s'' in group-1 can be chosen in $4C_2 = 6$ ways. In two cases, both s'' are at one of the ends of the broken line. Similarly, two s'' can be chosen in group-2 at the end of the broken line in two ways. When these s'' are connected together, one gets closed loops. In each case, the ends of the joining lines can be interchanged to give another way of connection (r.h.s of Figure 6.7). Thus a total of $2 \times 2 \times 2 = 8$ closed loops are obtained.

The contribution from any of the diagrams, say the first one, is

$$\begin{aligned} Dia-1 &= \sum_{ijlm} \sum_{k_1 k_2 k_7 k_8}^{B} s'_{i\mathbf{k}_1} s'_{i\mathbf{k}_2} s'_{m\mathbf{k}_7} s'_{m\mathbf{k}_8} \\ &\times \sum_{k_3 k_4 k_5 k_6}^{A} \delta(\mathbf{k}_3 + \mathbf{k}_5)\delta_{jl} G_0(k_3)\delta(\mathbf{k}_4 + \mathbf{k}_6)\delta_{jl} \\ &\times G_0(k_4)\delta(\mathbf{k}_1 + \cdots + \mathbf{k}_4)\delta(\mathbf{k}_5 + \cdots + \mathbf{k}_8). \end{aligned}$$

Noting that

$$\begin{aligned}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\delta(-\mathbf{k}_3-\mathbf{k}_4+\mathbf{k}_7+\mathbf{k}_8)\\ = \ \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_7+\mathbf{k}_8)\delta(-\mathbf{k}_3-\mathbf{k}_4+\mathbf{k}_7+\mathbf{k}_8),\end{aligned}$$

one gets

$$\begin{aligned}Dia-1 &= n\sum_{im}\sum_{k_1k_2k_7k_8}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}s'_{m\mathbf{k}_7}s'_{m\mathbf{k}_8}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_7+\mathbf{k}_8)\\ &\times \sum_{k_3k_4}^{A}\delta(-\mathbf{k}_3-\mathbf{k}_4+\mathbf{k}_7+\mathbf{k}_8)G_0(k_3)G_0(k_4). \qquad (6.9)\end{aligned}$$

Now, consider the case when two s'' in group-1 is at one end of the broken line and the two s'' in group-2 are at different ends of the broken line. The first arrangement can be done in 2 ways while in the second there are 4 ways. Thus there are $2\times4\times2=16$ (the last 2 is for interchange of ends of the joining lines) ways. In a similar way, the s'' in group-1 can be put at different ends of the broken line while the s'' in group-2 are at the same end of the broken line. This gives another 16 ways of making connections. Thus a total of 32 ways of connections are obtained. Eight of these cases are shown in Figure 6.8. Another 8 diagrams arise by choosing for s'' the solid lines at the left of the broken line of group-1. Then a similar 16 diagrams result with two s'' at the same end of the broken line of group-2.

Now the two s'' of group-1 and group-2 can be put at different ends of the broken lines. There are four ways of doing this in each group, thus $4\times4\times2=32$ (the last factor 2 is for interchange of ends of the joining lines) diagrams are obtained. Eight of these are shown in Figure 6.9. The remaining 24 comes from the other three ways of choosing s'' in group-1. The 64 diagrams (32+32) obtained are not closed loops and hence each yields a weight unity. The first diagram in Figure 6.9 yields

$$\begin{aligned}Dia-1 &= \sum_{ijlm}\sum_{k_1k_3k_5k_7}^{B} s'_{i\mathbf{k}_1}s'_{j\mathbf{k}_3}s'_{l\mathbf{k}_5}s'_{m\mathbf{k}_7}\\ &\times \sum_{k_2k_4k_6k_8}^{A}\delta(\mathbf{k}_2+\mathbf{k}_8)\delta_{im}G_0(k_2)\delta(\mathbf{k}_4+\mathbf{k}_6)\delta_{jl}\\ &\times G_0(k_4)\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\delta(\mathbf{k}_5+\cdots+\mathbf{k}_8)\\ &= \sum_{im}\sum_{k_1k_3k_5k_7}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_7}s'_{j\mathbf{k}_3}s'_{j\mathbf{k}_5}\delta(\mathbf{k}_1+\mathbf{k}_3+\mathbf{k}_5+\mathbf{k}_7)\\ &\times \sum_{k_2k_4}^{A}\delta(+\mathbf{k}_5+\mathbf{k}_7-\mathbf{k}_2-\mathbf{k}_4)G_0(k_2)G_0(k_4). \qquad (6.10)\end{aligned}$$

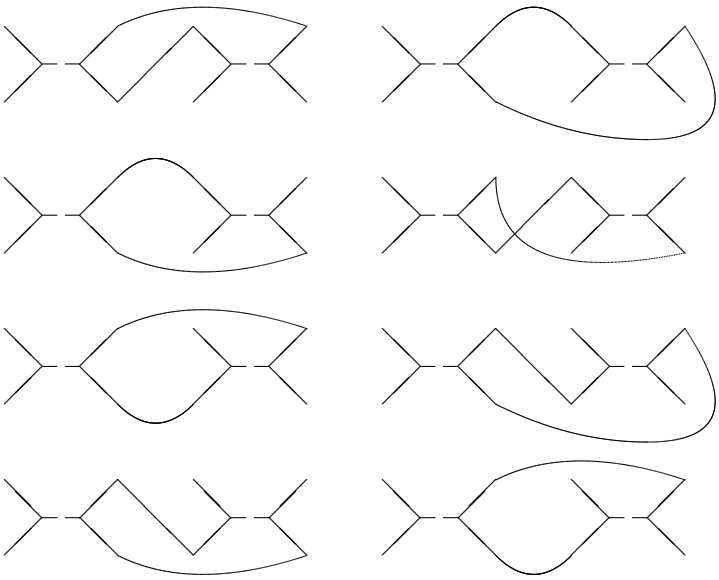

Figure 6.8: Diagrams in $< \widetilde{H}_1[s', s'']^2 >$.

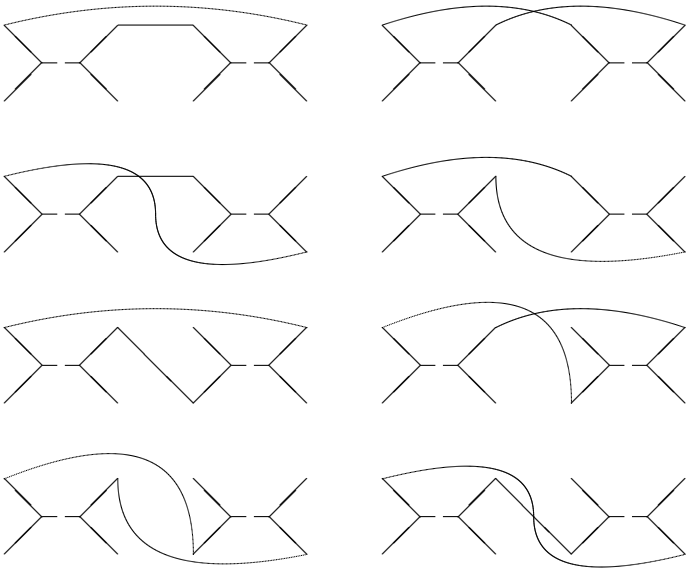

Figure 6.9: Diagrams in $< \widetilde{H}_1[s', s'']^2 >$.

Eqs.(6.9) and (6.10) are exactly of the same type. On substituting the factors $(8n+64)$ and $(u/8)^2L^{-2d}$, the contribution from the case $(N_1,N_2)=(2,2)$ is found to be

$$\begin{aligned}
&\frac{1}{2}\Big[< \widetilde{H}_1^2[s',s''] > - < \widetilde{H}_1[s',s''] >^2 \Big](with\ four\ s'\ terms) \\
&= \frac{1}{2}(\frac{u}{8})^2 L^{-2d}(8n+64)\sum_{ij}\sum_{k_1k_2k_3k_4}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}s'_{j\mathbf{k}_3}s'_{j\mathbf{k}_4} \\
&\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)v(\mathbf{k}_3,\mathbf{k}_4), \qquad (6.11)
\end{aligned}$$

where the term $v(\mathbf{k}_3,\mathbf{k}_4)$ is given by

$$v(\mathbf{k}_3,\mathbf{k}_4)=\sum_{k_5k_6}^{A} G_0(k_5)G_0(k_6)\delta(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6).$$

But for the dependence of v on $\mathbf{k}_3$ and $\mathbf{k}_4$ and the range of wave vectors (which are in the k_B group), the expression in Eq.(6.11) is identical to the quartic term in $H[s]$. To bring it to the required form, $v(k_3,\mathbf{k}_4)$ may be approximated by its value at $\mathbf{k}_3=\mathbf{k}_4=0$. That is,

$$v(\mathbf{k}_3,\mathbf{k}_4)\approx v(0,0)=\sum_{k_5}^{A} G_0^2(k_5).$$

Thus Eq.(6.11) reduces to

$$\begin{aligned}
&\frac{1}{2}\Big[< \widetilde{H}_1^2[s',s''] > - < \widetilde{H}_1[s',s''] >^2 \Big](with\ four\ s'\ terms) \\
&= \frac{1}{2}(\frac{u}{8})^2 L^{-2d}(8n+64)\sum_{ij}\sum_{k_1k_2k_3k_4}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}s'_{j\mathbf{k}_3}s_{j\mathbf{k}_4} \\
&\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\sum_{k_5}^{A} G_0^2(k_5). \qquad (6.12)
\end{aligned}$$

The exercise just completed has to be repeated for the case $(N_1,N_2)=(3,3)$.

Case $(N_1,N_2)=(3,3)$

Here there are three s'' in each group. As explained earlier, if two s'' of the same group are connected, then the wave vector sum of that group can not be zero. Hence every s'' of group-1 should be connected to a s'' of group-2. Three s'' of each group can be chosen in $4C_3=4$ ways. With a particular choice of s'' in group-1 and group-2, six ways of connection can be established. This includes the factor two arising out of the possibility of interchange of ends of the joining lines. Thus the total factor coming up from all ways of connections

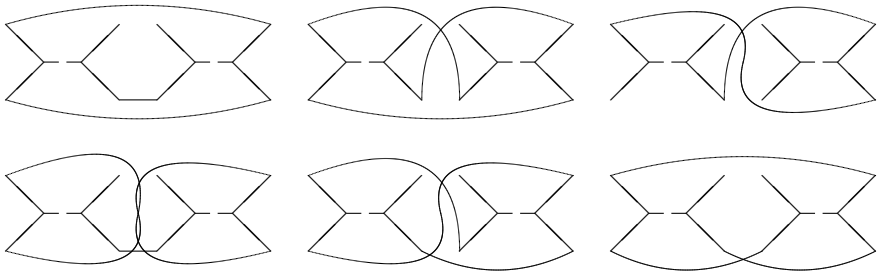

Figure 6.10: Diagrams in $< \widetilde{H}_1[s', s'']^2 >$.

is $4 \times 4 = 16$ times the factor obtained from a particular choice of s'' in group-1 and group-2. The diagrams resulting from a specific choice of s'' are shown in Figure 6.10. Here, there are two closed loops and four open loops thus giving a factor $(2n + 4)$. Then the total factor is $16(2n + 4)$. The first diagram of Figure 6.10 can be evaluated as

$$\begin{aligned} Dia-1 &= \sum_{ijlm} \sum_{k_4 k_6}^{B} s'_{j\mathbf{k}_4} s'_{l\mathbf{k}_6} \sum_{k_1 k_2 k_3 k_5 k_7 k_8}^{A} \delta(\mathbf{k}_2 + \mathbf{k}_8)\delta_{im} G_0(k_2) \\ &\times \delta(\mathbf{k}_3 + \mathbf{k}_5)\delta_{jl} G_0(k_3)\delta(\mathbf{k}_1 + \mathbf{k}_7)\delta_{im} \\ &\times G_0(k_1)\delta(\mathbf{k}_1 + \cdots + \mathbf{k}_4)\delta(\mathbf{k}_5 + \cdots + \mathbf{k}_8) \\ &= n \sum_{j} \sum_{k_4}^{B} |s'_{j\mathbf{k}_4}|^2 \sum_{k_1 k_2 k_3}^{A} G_0(k_1) G_0(k_2) G_0(k_3) \\ &\times \delta(\mathbf{k}_4 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \end{aligned}$$

Defining $w(\mathbf{k}_4)$ as

$$\begin{aligned} w(\mathbf{k}_4) &= \sum_{k_1 k_2 k_3} G_0(k_1) G_0(k_2) G_0(k_3)\delta(\mathbf{k}_4 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ &= \sum_{k_1 k_2}^{A} G_0(k_1) G_0(k_2) G_0(\mathbf{k}_4 + \mathbf{k}_1 + \mathbf{k}_2), \end{aligned}$$

one gets

$$\begin{aligned} &\frac{1}{2}\Big[< \widetilde{H}_1^2[s', s''] > - < \widetilde{H}_1[s', s''] >^2 \Big] \ (with\ two\ s'\ terms) \\ &= \frac{1}{2}(\frac{u}{8})^2 L^{-2d} 16(2n+4) \sum_{i} \sum_{k_1}^{B} |s'_{i\mathbf{k}_1}|^2 w(k_1). \end{aligned}$$

Since $G_0(k_1) = (r + ck^2)^{-1}$, w depends only on $|\mathbf{k}_1|$. Further, $w(\mathbf{k}_1)$ is an even function of $\mathbf{k}_1$ as it is unaltered by changing $\mathbf{k}_1$ to $-\mathbf{k}_1$. Hence it may be

$\widetilde{H}_1[s', s'']H_1[0, s''] \; =$

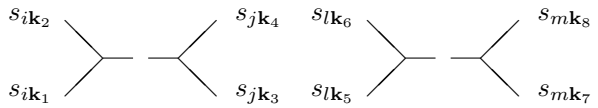

Figure 6.11: Diagram for $\widetilde{H}_1[s', s'']H_1[0, s'']$.

approximated as

$$w(\mathbf{k}_1) \approx w(0) - k_1^2 w^*,$$

where $w(0)$ is given by

$$w(0) = \sum_{k_2 k_3}^{A} G_0(k_2) G_0(k_3) G_0(\mathbf{k}_2 + \mathbf{k}_3),$$

and w^* is the coefficient of $-k_1^2$ in $[w(k_1) - w(0)]$. Thus with this approximation for $w(k_1)$, one gets

$$\begin{aligned} &\frac{1}{2}\Big[< \widetilde{H}_1^2[s', s''] > - < \widetilde{H}_1[s', s''] >^2 \Big] (with\ two\ s'\ terms) \\ &= \frac{1}{2}(\frac{u}{8})^2 L^{-2d} 16(2n+4) \sum_i \sum_{k_1}^{B} |s'_{i\mathbf{k}_1}|^2 [w(0) - k_1^2 w^*]. \end{aligned} \tag{6.13}$$

What remains to be computed is the last term in Eq.(6.6).

(B). $< \widetilde{H}_1[s', s'']H_1[0, s''] > - < \widetilde{H}_1[s', s''] >< H_1[0, s''] >$

Recall that $\widetilde{H}_1[s', s'']$ is represented as a diagram with four solid lines and a broken line and there should be at least one s' and one s'' in it. $H_1[0, s'']$ also can be represented using the same diagram, but now all the solid lines should denote s''. Thus the diagram for their product is as shown in Figure 6.11. To find the average, all the solid lines representing s'' should be connected. If there is only one s'' or three s'' in $\widetilde{H}_1[s', s'']$, then one of them remains unconnected and the contribution to the average from such a case is zero. Hence the cases, when there are two s'' in $\widetilde{H}_1[s', s'']$, alone are important. Then there are only two s' terms and hence the contribution is to the quadratic term in $H'[s]$.

The two s'' of $\widetilde{H}_1[s', s'']$ and the four s'' of $H_1[0, s'']$ can be connected among themselves. Note that there are $4C_2 = 6$ ways of connecting the two s'' of $\widetilde{H}_1[s', s'']$ and three ways of connecting the s'' of $H_1[0, s'']$. The various diagrams arising in this way are represented in Figure 6.12. These diagrams are

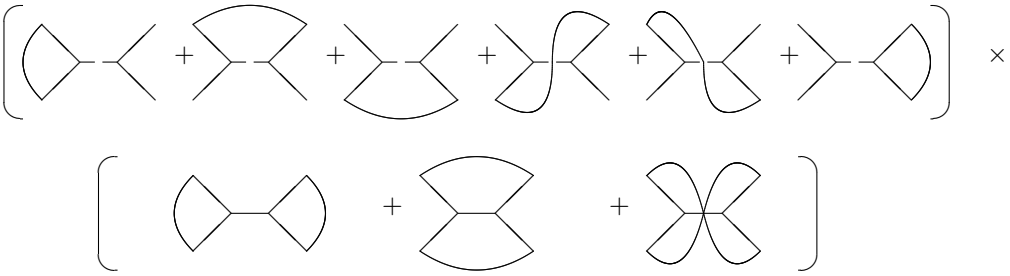

Figure 6.12: Diagram for $< \widetilde{H}_1[s', s''] >< H_1[0, s''] >$.

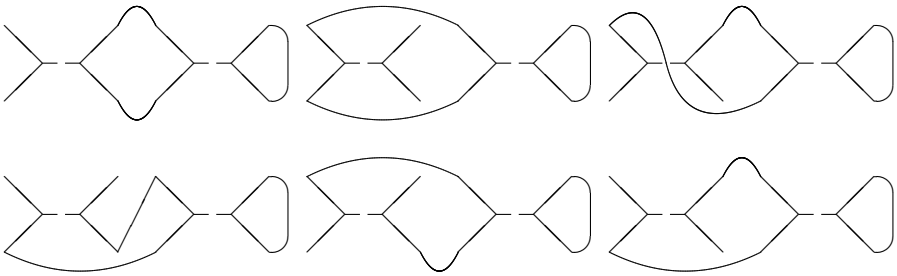

Figure 6.13: Diagram for $< \widetilde{H}_1[s', s'']H_1[0, s''] >$.

just those obtained from a calculation of $< \widetilde{H}_1[s', s''] >< H_1[0, s''] >$. Hence the contribution to $< \widetilde{H}_1[s', s'']H_1[0, s''] > - < \widetilde{H}_1[s', s''] >< H_1[0, s''] >$ arises only by connecting s'' of $\widetilde{H}_1[s', s'']$ with those of $H_1[0, s'']$. Notice that two s'' of $H_1[0, s'']$ always have to be connected among themselves. First of all, connect the two s'' lines at one end of the broken line of $H_1[0, s'']$. The resulting diagrams are shown in Figure 6.13. The weight of the first two is n^2 each while others have a weight of n each. Thus the total weight is $2n^2 + 4n$. Exactly the same thing can be done by connecting the s'' lines on the other side of the broken line in $H_1[0, s'']$. Then another factor of $2n^2+4n$ is obtained.

Now, consider the case of connecting two s'' of $H_1[0, s'']$, one from each end of the broken line. This can be done in four ways. On connecting the s'' lines above the broken line, the diagrams in Figure 6.14 are obtained. Thus two closed loops and four open loops, giving a factor $2n + 4$, are obtained. Exactly the same factor $2n + 4$ arises by connecting s'' lines below the broken line. Now connect the s'' above the broken line to the one below. This yields

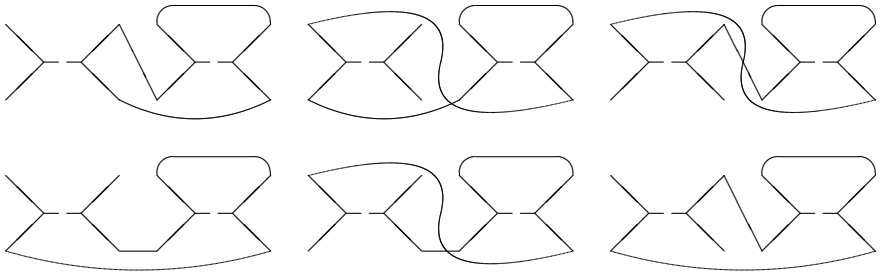

Figure 6.14: Diagram for $< \widetilde{H}_1[s', s'']H_1[0, s''] >$.

the diagrams of Figure 6.15 which again give the factor $2n + 4$. One more possibility of similar connection yields another factor $2n+4$. Thus the weight obtained is $2(2n^2+4n)+4(2n+4) = 4(n+2)^2$. However, an additional factor 2 is found by interchanging the ends of the joining lines in each of the diagrams. Hence the total weight is $8(n + 2)^2$. The first diagram from Figure 6.15 can be evaluated as

$$\begin{aligned} Dia-1 &= \sum_{ijlm}\sum_{k_1k_2}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}\sum_{k_3k_4}^{A}\sum_{k_5k_6k_7k_8}^{A}\delta(\mathbf{k}_3+\mathbf{k}_5)\delta_{jl}G_0(k_3) \\ &\times \delta(\mathbf{k}_4+\mathbf{k}_8)\delta_{jm}G_0(k_4)\delta(\mathbf{k}_6+\mathbf{k}_7) \\ &\times \delta_{lm}G_0(k_6)\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\delta(\mathbf{k}_5+\cdots+\mathbf{k}_8) \\ &= n\sum_{i}\sum_{k_1k_2}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}\sum_{k_3k_4}^{A}\sum_{k_6}^{A}G_0(k_3)G_0(k_4)G_0(k_6) \\ &\times \delta(-\mathbf{k}_3+\mathbf{k}_6-\mathbf{k}_6-\mathbf{k}_4) \\ &= n\sum_{i}\sum_{k_1}^{B}|s'_{ik_1}|^2\sum_{k_3}^{A}G_0(k_3)^2\sum_{k_6}^{A}G_0(k_6). \end{aligned}$$

This result yields

$$\begin{aligned} &< \widetilde{H}_1[s', s'']H_1[0, s''] > - < \widetilde{H}_1[s', s''] >< H_1[0, s''] > \\ &= (\frac{u}{8})^2 L^{-2d} 8(n+2)^2 \sum_{i}\sum_{k_1}^{B}|s'_{i\mathbf{k}_1}|^2\sum_{k_2}^{A}G_0^2(k_2)\sum_{k_3}^{A}G_0(k_3). \end{aligned} \tag{6.14}$$

Final Result

Collecting together the results of Eqs.(6.12) ($\mathbf{s'}^{\,4}$ term), (6.13) ($\mathbf{s'}^{\,2}$ term)

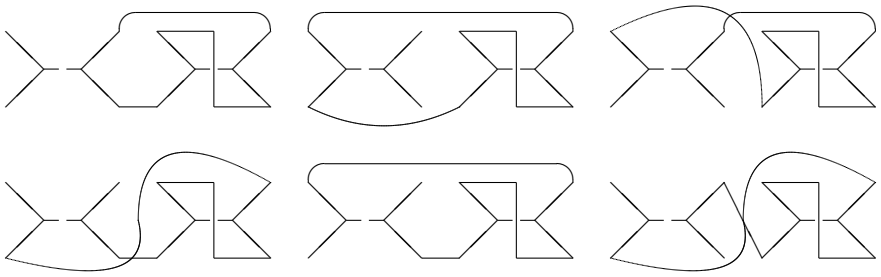

Figure 6.15: Diagram for $< \widetilde{H}_1[s', s'']H_1[0, s''] >$.

and (6.14) ($\mathbf{s'}^{\,2}$ term) the second order term is obtained as

$$\begin{aligned}
2^{nd}\ order\ term &= \frac{1}{2}(\frac{u}{8})^2 L^{-2d}(8n+64)\sum_{ij}\sum_{k_1k_2k_3k_4}^{B} s'_{i\mathbf{k}_1}s'_{i\mathbf{k}_2}s'_{j\mathbf{k}_3}s'_{j\mathbf{k}_4} \\
&\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\sum_{k_5}^{A} G_0^2(k_5) \\
&+ \frac{1}{2}(\frac{u}{8})^2 L^{-2d}16(2n+4)\sum_{i}\sum_{k_1}^{B}|s'_{i\mathbf{k}_1}|^2[w(0)-k_1^2w^*] \\
&+ (\frac{u}{8})^2 L^{-2d}8(n+2)^2\sum_{i}\sum_{k_1}^{B}|s'_{i\mathbf{k}_1}|^2\sum_{k_2}^{A}G_0^2(k_2)\sum_{k_3}^{A}G_0(k_3).
\end{aligned}$$

The last two steps of RG are effected by the replacements

$$s'_{i\mathbf{k}} \to s_{i\mathbf{k}'}q^{1-\eta/2}, \mathbf{k}' = q\mathbf{k}, L' = L/q.$$

The factor L^{-d} is absorbed in the summation, $\sum_k^A(\cdots)$, since the continuum limit is to be taken later. Thus one gets

$$\begin{aligned}
2^{nd}\ order\ term &= \frac{u}{8}^2 L'^{\,-d}\frac{n+8}{2}q^{4-d-2\eta}\sum_{ij}\sum_{\{k_m\}\le\Lambda} s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4} \\
&\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)[L^{-d}\sum_{k_5}^{A}G_0^2(k_5)] \\
&+ \frac{u^2}{2}\frac{n+2}{2}q^{2-\eta}\sum_{i}\sum_{k_1\le\Lambda}|s_{i\mathbf{k}_1}|^2\{L^{-2d}[w(0)-k_1^2w^*]\} \\
&+ \frac{u^2}{2}\frac{(n+2)^2}{4}q^{2-\eta}\sum_{i}\sum_{k_1\le\Lambda}|s_{i\mathbf{k}_1}|^2 \\
&\times [L^{-d}\sum_{k_2}^{A}G_0^2(k_2)][L^{-d}\sum_{k_3}^{A}G_0(k_3)]. \qquad (6.15)
\end{aligned}$$

Subtraction of Eq.(6.15) from Eq.(6.8) yields

$$\begin{aligned}
H'[s] &= \frac{1}{2}\sum_{i}\sum_{k\le\Lambda}(rq^{2-\eta}+cq^{-\eta}k^2)|s_{i\mathbf{k}}|^2 \\
&+ \frac{u}{8}(2n+4)\sum_{i}\sum_{k\le\Lambda}|s_{i\mathbf{k}}|^2 q^{2-\eta}[L^{-d}\sum_{ik_1}^{A}G_0(k_1)] \\
&- \frac{u^2}{2}\frac{n+2}{2}q^{2-\eta}\sum_{i}\sum_{k_1\le\Lambda}|s_{i\mathbf{k}_1}|^2\{L^{-2d}[w(0)-k_1^2 w^*]\} \\
&- \frac{u^2}{2}\frac{(n+2)^2}{4}q^{2-\eta}\sum_{i}\sum_{k_1\le\Lambda}|s_{i\mathbf{k}_1}|^2[L^{-d}\sum_{k_2}^{A}G_0(k_2)^2][L^{-2d}\sum_{k_3}^{A}G_0(k_3)] \\
&+ [\frac{u}{8}-\frac{u^2}{8}\frac{n+8}{2}]L'^{\,-d}q^{4-d-2\eta}\sum_{ij}\sum_{\{k_m\}\le\Lambda}s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4} \\
&\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)[L^{-d}\sum_{k_5}^{A}G_0^2(k_5)]+O(u^3).
\end{aligned}$$

It can be rewritten as

$$\begin{aligned}
H'[s] &= \frac{1}{2}\sum_{i}\sum_{k\le\Lambda}(r'+ck^2)|s_{i\mathbf{k}}|^2 \\
&+ \frac{u'}{8}L'^{\,-d}\sum_{ij}\sum_{\{k_m\}\le\Lambda}s_{i\mathbf{k}_1}s_{i\mathbf{k}_2}s_{j\mathbf{k}_3}s_{j\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)+O(u^3),
\end{aligned}$$

where the new parameters are defined as

$$\begin{aligned}
r' &= q^{2-\eta}\Big[r+\frac{u}{2}(n+2)\{L^{-d}\sum_{k_1}^{A}G_0(k_1)\}-\frac{u^2}{2}(n+2)\{L^{-2d}w(0)\} \\
&- \frac{u^2}{4}(n+2)^2\{L^{-d}\sum_{k_1}^{A}G_0^2(k_1)\}\{L^{-d}\sum_{k_2}^{A}G_0(k_2)\}\Big]+O(u^3). \\
u' &= q^{4-d-2\eta}\Big[u-\frac{u^2}{2}(n+8)\{L^{-d}\sum_{k_5}^{A}G_0^2(k_5)\}\Big]+O(u^3), \\
c' &= q^{-\eta}\Big[c+\frac{u^2}{2}(n+2)\{L^{-2d}w^*\}\Big]+O(u^3).
\end{aligned}$$

These equations define the recursion relations to second order accuracy. Now, the continuum limit is taken by replacing $L^{-d}\sum_k^A(\cdots)$ with the integral $(2\pi)^{-d}\int^A(\cdots)d\mathbf{k}$. The recursion relations then become

$$\begin{aligned}
r' &= q^{2-\eta}\Big[r+\frac{u}{2}(n+2)(2\pi)^{-d}\int^{A}G_0(k)d\mathbf{k}-u^2D\Big]+O(u^3), \\
u' &= q^{4-d-2\eta}\Big[u-\frac{u^2}{2}(n+8)(2\pi)^{-d}\int^{A}G_0^2(k)d\mathbf{k}\Big]+O(u^3),
\end{aligned}$$

$$c' = cq^{-\eta}\Big[1 + u^2 \ln(q) E\Big] + O(u^3),$$

where D is defined as

$$\begin{aligned} D &= \frac{n+2}{2}(2\pi)^{-2d} \int^A d\mathbf{k}_1 \int^A d\mathbf{k}_2 G_0(k_1) G_0(k_2) G_0(\mathbf{k}_1 + \mathbf{k}_2) \\ &+ \frac{(n+2)^2}{4}(2\pi)^{-d} \int^A d\mathbf{k}_1 G_0^2(k_1)(2\pi)^{-d} \int^A d\mathbf{k}_2 G_0(k_2). \end{aligned}$$

In obtaining this relation, the definition of $w(0)$ has been used. Further, in the expression for c', E has been defined as

$$E = \frac{n+2}{2c\ln(q)} \frac{w^*}{L^{2d}}.$$

Later it is shown that for large values of q (which is arbitrary), E is independent of q. Since u is assumed to be small, c' can be rewritten as

$$c' = cq^{-\eta} q^{u^2 E} + O(u^3).$$

6.4 Fixed Points of recursion Relations

The recursion relations derived in the previous section are similar to those obtained using Wilson's recursion formulae. Only one assumption, namely, the parameter u is small, has been employed in the present derivation.

(i) Now, the parameter η can be chosen as $\eta = u^2 E$. Then there is a fixed point $\mu^* = (r^*, u^*, c^*) = (0, 0, c^*)$ where c^* is arbitrary. Earlier it was shown that this fixed point has appropriate properties for $d > 4$. In fact, one finds that $\eta = 0$ since $u^* = 0$. Then, linearising about μ^*, one gets

$$\begin{aligned} \Delta r' &= q^2\Big[\Delta r + \Delta u \frac{n+2}{2}(2\pi)^{-d} \int^A G_0(k) d\mathbf{k}\Big], \\ \Delta u' &= q^{\epsilon} \Delta u, \quad \epsilon = 4 - d. \end{aligned}$$

They yield $y_1 = 2$ and $y_2 = \epsilon$ and hence the Landau's theory exponents for $d > 4$. This fixed point is not appropriate for $d \leq 4$.

(ii) Having fixed $\eta = u^2 E$, the recursion relations to be considered are

$$\begin{aligned} r' &= q^{2-\eta}\Big[r + \frac{u}{2}(n+2)(2\pi)^{-d} \int^A G_0(k) d\mathbf{k} - u^2 D\Big] + O(u^3), \\ u' &= q^{4-d-2\eta}\Big[u - u^2 \frac{n+8}{2}(2\pi)^{-d} \int^A G_0^2(k) d\mathbf{k}\Big] + O(u^3). \end{aligned}$$

The second equation yields a fixed point relation

$$1 = q^{4-d-2\eta}\Big[1 - u^* \frac{n+8}{2}(2\pi)^{-d} \int^A G_0^2(k) d\mathbf{k}\Big] + O(u^{*\,2}).$$

Introducing the notation, κ_d=(area of unit sphere)$\times(2\pi)^{-d}$, u^* is found to be

$$u^* = (1-q^{-\epsilon+2\eta})\frac{2}{n+8}\kappa_d^{-1}\Big[\int\limits_{\Lambda/q}^{\Lambda}\frac{k^{d-1}dk}{(r^*+ck^2)^2}\Big]^{-1} + O(u^{*\,2}).$$

The equation for r' shows that

$$(1-q^{2-\eta})r^* \;=\; q^{2-\eta}\frac{u^*}{2}(n+2)\kappa_d\int\limits_{\Lambda/q}^{\Lambda}\frac{k^{d-1}dk}{r^*+ck^2} + O(u^{*\,2}).$$

This means that the term $-u^2D$ in r' does not contribute to the fixed point value of r^* accurate to $O(u^2)$. Further, note that the value of r^* is of the order of u^*. As discussed earlier, u^* and r^* should be small since they have been obtained by a perturbation expansion. So $(r^*+ck^2)^{-1}$ can be expanded in powers of r^*. Keeping terms of $O(u^*) = O(r^*)$, one gets

$$\begin{aligned}
u^* &= (1-q^{-\epsilon+2\eta})\frac{2}{n+8}\frac{1}{\kappa_d}\Big[\int\limits_{\Lambda/q}^{\Lambda}\frac{k^{d-1}dk}{c^2k^4} + O(r^*)\Big]^{-1} + O(u^{*\,2}) \\
&= (1-q^{-\epsilon+2\eta})\frac{2}{n+8}\frac{c^2}{\kappa_d}\Big[\frac{\Lambda^{d-4}}{d-4}(1-q^{4-d}\,) + O(r^*)\Big]^{-1} + O(u^{*\,2}) \\
&= \epsilon(q^{-\epsilon+2\eta}-1)\frac{2}{n+8}\frac{c^2}{\kappa_d}\Lambda^{4-d}\Big[(1-q^{\epsilon}) + O(\epsilon r^*)\Big]^{-1} + O(u^{*\,2}).
\end{aligned}$$

Since $u^* \sim \epsilon$, it can be concluded that the perturbation expansion is meaningful only for small values of ϵ. Therefore taking $u^* \sim r^* \sim O(\epsilon)$ one finds that

$$u^* = \epsilon(q^{-\epsilon+2\eta}-1)\frac{2}{n+8}\frac{c^2}{\kappa_d}\frac{\Lambda^{4-d}}{1-q^{\epsilon}} + O(\epsilon^2).$$

The terms κ_d and Λ^{4-d} (which depend on d) can be calculated at $\epsilon = 0$, i.e. at $d = 4$. Since $\eta \sim O(u^{*\,2}) \sim O(\epsilon^2)$, u^* reduces to

$$\begin{aligned}
u^* &= \epsilon q^{-\epsilon}\frac{2}{n+8}\frac{c^2}{\kappa_4} + O(\epsilon^2) \\
&= \epsilon c^2\frac{2}{n+8}\frac{1}{\kappa_4} + O(\epsilon^2).
\end{aligned}$$

Though the calculations were accurate to $O(u^2) \sim O(\epsilon^2)$, the value of u^* has accuracy of $O(\epsilon)$ only. The exponent η is given by

$$\eta = u^{*\,2}E \;=\; \epsilon^2\Big[c^2\frac{2}{n+8}\frac{1}{\kappa_4}\Big]^2E + O(\epsilon^3).$$

Note that $\eta \sim O(\epsilon^2)$ and hence this exponent is relatively small. Now, since $\eta \sim O(\epsilon^2)$, the value of r^* can be written as

$$r^* = \frac{q^2}{1-q^2}\frac{u^*}{2}(n+2)\frac{\kappa_d}{c}\int\limits_{\Lambda/q}^{\Lambda} k^{d-3}dk + O(\epsilon^2).$$

All terms can be evaluated at $d = 4$ since $u^* \sim O(\epsilon)$. Then

$$\begin{aligned} r^* &= \frac{q^2}{1-q^2}\frac{u^*}{2}(n+2)\frac{\kappa_4}{c}\Lambda^2\frac{1-q^{-2}}{2} + O(\epsilon^2) \\ &= -\frac{u^*}{2}(n+2)\frac{\kappa_4}{c}\frac{\Lambda^2}{2} + O(\epsilon^2) \\ &= -\epsilon c\frac{n+2}{n+8}\frac{\Lambda^2}{2} + O(\epsilon^2). \end{aligned}$$

Having obtained the fixed point values and η, the transformation equations can be linearised to obtain the remaining exponents.

6.5 Exponents Accurate to $O(\epsilon)$

In deriving the fixed points of the recursion relations, it is found that r^* and u^* are of order $\epsilon = 4 - d$. Therefore, the recursion relations can be simplified by expanding $G_0(k)$ in powers of r and keeping terms like $O(u^2), O(ru)$, etc. In fact the implicit assumption is that $r \sim u \sim O(\epsilon)$ and terms up to $O(\epsilon^2)$ are to be retained. Since $\eta \sim O(\epsilon^2)$, the term $q^{-\eta}$ can be omitted from the recursion relations. Now,

$$\begin{aligned} \frac{1}{(2\pi)^d}\int\limits_{\Lambda/q}^{\Lambda} G_0(k)d\mathbf{k} &= (2\pi)^{-d}\int\limits_{\Lambda/q}^{\Lambda}[\frac{1}{ck^2} - \frac{r}{c^2k^4}]d\mathbf{k} + O(r^2) \\ &= \frac{\kappa_d}{c}\Big[\frac{\Lambda^{d-2}}{d-2}(1-q^{2-d}) - \frac{r}{c}\frac{\Lambda^{d-4}}{d-4}(1-q^{4-d}\ )\Big] + O(r^2). \end{aligned}$$

In a similar way

$$\frac{1}{(2\pi)^d}\int\limits_{\Lambda/q}^{\Lambda} G_0^2(k)d\mathbf{k} = \frac{\kappa_d}{c^2}\frac{\Lambda^{d-4}}{d-4}(1-q^{4-d}\ ) + O(r^2).$$

Hence the recursion relations become

$$\begin{aligned} r' &= q^2\Big[r + \frac{u}{2}(n+2)\frac{\kappa_d}{c}\{\frac{\Lambda^{d-2}}{d-2}(1-q^{2-d}) \\ &- \frac{r}{c}\frac{\Lambda^{d-4}}{d-4}(1-q^{4-d}\ )\} - u^2D_0\Big] + O(u^3), \\ u' &= q^\epsilon\Big[u - \frac{u^2}{2}(n+8)\frac{\kappa_d}{c^2}\frac{\Lambda^{d-4}}{d-4}(1-q^{4-d}\ )\Big] + O(u^3), \end{aligned}$$

where D_0 is the value of D at $r = 0$. Now, note that

$$\begin{aligned}
\kappa_d \frac{\Lambda^{d-4}}{d-4}(1 - q^{4-d}\ ) &= \kappa_d \frac{\Lambda^{-\epsilon}}{\epsilon}(q^\epsilon - 1) = \kappa_4 \ln(q) + O(\epsilon), \\
\kappa_d \frac{\Lambda^{d-2}}{d-2}(1 - q^{2-d}) &= \kappa_4 \frac{\Lambda^2}{2}(1 - q^{-2}) + C\epsilon + O(\epsilon^2),
\end{aligned}$$

where C is some constant. The constant D_0 can also be evaluated at $d = 4$ and it can be written as $D_0 = D_{00} + O(\epsilon)$. Then, keeping terms up to $O(\epsilon^2) = O(u^2) = O(ru)$, one finds

$$\begin{aligned}
r' &= q^2\Big[r + \frac{u}{2}(n+2)\{\kappa_4 \frac{\Lambda^2}{2c}(1 - q^{-2}) \\
&+ C\epsilon - r\frac{\kappa_4}{c^2}\ln(q)\} - u^2 D_{00}\Big] + O(\epsilon^3), \qquad (6.16) \\
u' &= q^\epsilon\Big[u - \frac{u^2}{2}(n+8)\frac{\kappa_4}{c^2}\ln(q)\Big] + O(\epsilon^3). \qquad (6.17)
\end{aligned}$$

Note that these recursion relations yield the fixed points, r^* and u^* accurate to $O(\epsilon)$, obtained earlier. For example,

$$\begin{aligned}
u^* &= \frac{2c^2}{\kappa_4}\frac{1 - q^{-\epsilon}}{(n+8)\ln(q)} + O(\epsilon^2) \\
&= \frac{2c^2}{\kappa_4}\frac{\epsilon}{n+8} + O(\epsilon^2),
\end{aligned}$$

and therefore

$$\begin{aligned}
(1 - q^2)r^* &= \frac{u^*}{2}(n+2)\kappa_4 \frac{\Lambda^2}{2c}(1 - q^2) + O(\epsilon^2), \\
r^* &= -\epsilon\frac{c}{2}\Lambda^2\frac{n+2}{n+8} + O(\epsilon^2).
\end{aligned}$$

The transformation equations can be linearised by writing $u' = u^* + \Delta u'$ etc. On keeping terms up to Δu, Eq.(6.17) yields

$$u^* + \Delta u' = q^\epsilon\Big[u^* + \Delta u - ({u^*}^2 + 2u^*\Delta u)\frac{n+8}{2}\frac{\kappa_4}{c^2}\ln(q)\Big] + O(\epsilon^3).$$

That is

$$\begin{aligned}
\Delta u' &= q^\epsilon\Big[1 - 2u^*\frac{n+8}{2}\frac{\kappa_4}{c^2}\ln(q)\Big]\Delta u + O(\epsilon^2) \\
&= q^\epsilon\Big[1 - 2\epsilon\ln(q)\Big]\Delta u + O(\epsilon^2). \qquad (6.18)
\end{aligned}$$

The coefficient of Δu (on the r.h.s) is accurate to $O(\epsilon)$ only. Thus the eigenvalues of the linearised transformation and hence the exponents are also accurate

to $O(\epsilon)$ only. Eq.(6.16) shows that

$$\begin{aligned} r^* + \Delta r &= q^2\Big[r^* + \Delta r + (u^* + \Delta u)\frac{n+2}{2}\{\kappa_4\frac{\Lambda^2}{2c}(1-q^{-2}) + C\epsilon \\ &- \frac{\kappa_4}{c^2}(r^* + \Delta r)\ln(q)\} - (u^{*\,2} + 2u^*\Delta u)D_0\Big] + O(\epsilon^3). \end{aligned}$$

Keeping terms linear in Δr and Δu, one gets

$$\begin{aligned} \Delta r' &= q^2\Big[\Delta r\{1 - u^*\frac{n+2}{2}\frac{\kappa_4}{c^2}\ln(q)\} + \Delta u\{\frac{n+2}{2}\kappa_4\frac{\Lambda^2}{2c}(1-q^{-2}) \\ &+ C\epsilon - r^*\frac{n+2}{2}\frac{\kappa_4}{c^2}\ln(q) - 2u^*D_0\}\Big] + O(\epsilon^2). \end{aligned}$$

This relation also is accurate to $O(\epsilon)$ since u^* has that accuracy. Substitution of u^* and r^* yields

$$\begin{aligned} \Delta r' &= q^2[1 - \epsilon\ln(q)\frac{n+2}{n+8}]\Delta r \\ &+ [\frac{n+2}{2}\frac{\kappa_4}{2c}\Lambda^2(q^2-1) + O(\epsilon)]\Delta u + O(\epsilon^2). \end{aligned} \tag{6.19}$$

To first order accuracy in ϵ, Eqs.(6.18) and (6.19) can be written as

$$\begin{aligned} \Delta r' &= q^{2-\epsilon(n+2)/(n+8)}\Delta r + [B + O(\epsilon)]\Delta u + O(\epsilon^2), \\ \Delta u' &= q^\epsilon q^{-2\epsilon}\Delta u = q^{-\epsilon}\Delta u + O(\epsilon^2), \end{aligned}$$

where

$$B = \frac{n+2}{2}\kappa_4\frac{\Lambda^2}{2c}(q^2-1).$$

The linearised RG matrix is therefore given by

$$\widetilde{\mathbf{R}}_q = \begin{bmatrix} q^{2-\epsilon\ (n+2)/(n+8)} & B + O(\epsilon) \\ 0 & q^{-\epsilon} \end{bmatrix}.$$

Since the matrix is triangular, the eigenvalues are

$$\begin{aligned} \rho_1 &= q^{y_1} = q^{2-\epsilon(n+2)/(n+8)}, \\ \rho_2 &= q^{y_2} = q^{-\epsilon}. \end{aligned}$$

Hence y_1 and y_2 are given by

$$\begin{aligned} y_1 &= 2 - \epsilon\frac{n+2}{n+8}, \\ y_2 &= -\epsilon. \end{aligned}$$

Thus y_1 is positive and y_2 is negative for $d < 4$. Hence for $d < 4$, the fixed point has the desired property. The parameter y_1 is accurate to $O(\epsilon)$ and all

the exponents can be computed using the scaling laws to this accuracy. Since η is proportional to ϵ^2, it should be taken as zero at this level of approximation. The various exponents are

$$\begin{aligned}
\eta &= O(\epsilon^2), \\
\nu &= \frac{1}{2} + \frac{\epsilon}{4}\frac{n+2}{n+8} + O(\epsilon^2), \\
\gamma &= 1 + \frac{\epsilon}{2}\frac{n+2}{n+8} + O(\epsilon^2), \\
\alpha &= \frac{\epsilon}{2}\frac{4-n}{n+8} + O(\epsilon^2), \\
\beta &= \frac{1}{2} - \frac{3}{2}\frac{\epsilon}{n+8} + O(\epsilon^2), \\
\delta &= 3 + \epsilon + O(\epsilon^2).
\end{aligned}$$

These results reduce to those obtained with Wilson's recursion formulae when $n = 1$. Now, by evaluating the constant E, η can be obtained to second order accuracy.

6.6 Calculation of η to $O(\epsilon^2)$

The expression for the exponent η obtained earlier is $\eta = u^{*\,2}E$. Substituting for u^* one gets,

$$\eta = \epsilon^2\Big[\frac{2}{\kappa_4}\frac{c^2}{n+8}\Big]^2 E,$$

where E is given by

$$E = \frac{n+2}{2c\ln(q)}\frac{w^*}{L^{2d}}.$$

Therefore η becomes

$$\eta = \epsilon^2\Big[\frac{2}{\kappa_4}\frac{c^2}{n+8}\Big]^2 \frac{n+2}{2c\ln(q)}\frac{w^*}{L^{2d}}.$$

Note that w^* is proportional to the coefficient of $-k_1^2$ in $w(k_1)$. Therefore the expression

$$\begin{aligned}
&L^{-2d}[w(k_1) - w(0)] \\
&\quad = L^{-2d}\sum_{k_2 k_3}^{A} G_0(k_2)G_0(k_3)\Big[G_0(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) - G_0(\mathbf{k}_2+\mathbf{k}_3)\Big],
\end{aligned}$$

can be calculated to $O(k_1^2)$, and then the coefficient of $-k_1^2$ can be obtained. Since $\eta \sim O(\epsilon^2)$, it is sufficient to evaluate w^* at $d = 4$ and $r^* = 0$. Further,

taking the continuum limit with the replacement

$$L^{-d}\sum_{k}^{A}(\cdots) \rightarrow (2\pi)^{-d}\int_{\Lambda/q}^{\Lambda}(\cdots)d\mathbf{k},$$

the integral to be computed is

$$\begin{aligned} & I(k_1) \\ &= \frac{1}{(2\pi)^8 c^3}\int_{\Lambda/q}^{\Lambda} d\mathbf{k}_2 \int_{\Lambda/q}^{\Lambda} d\mathbf{k}_3\Big[k_2^{-2}k_3^{-2}\{(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)^{-2}-(\mathbf{k}_2+\mathbf{k}_3)^{-2}\}\Big]. \end{aligned}$$

Then w^* is given by

$$\frac{w^*}{L^{2d}} = -\lim_{k_1^2\rightarrow 0}\frac{I(k_1)}{k_1^2}.$$

Consider the integral over $\mathbf{k}_3$,

$$\begin{aligned} J_1(k_2) &= \frac{1}{(2\pi)^4}\int_{\Lambda/q}^{\Lambda} d\mathbf{k}_3 k_3^{-2}(\mathbf{k}_2+\mathbf{k}_3)^{-2} \\ &= \frac{1}{(2\pi)^4}\int_{\Lambda/q}^{\Lambda} dk_3 k_3^3 \int_0^{\pi}\sin^2(\theta_1)d\theta_1 \int_0^{\pi}\sin(\theta_2)d\theta_2 \\ &\times \int_0^{2\pi} d\phi\Big[k_3^{-2}\{k_2^2+k_3^2+2k_2k_3\cos(\theta_1)\}^{-1}\Big]. \end{aligned}$$

Here, the polar coordinates in four dimension, defined as

$$\begin{aligned} k_{31} &= k_3\cos(\theta_1), \\ k_{32} &= k_3\sin(\theta_1)\cos(\theta_2), \\ k_{33} &= k_3\sin(\theta_1)\sin(\theta_2)\cos(\phi), \\ k_{34} &= k_3\sin(\theta_1)\sin(\theta_2)\sin(\phi), \end{aligned}$$

have been used. k_{3i} $(1 \leq i \leq 4)$ are the cartesian components of $\mathbf{k}_3$ and θ_1 is the angle between $\mathbf{k}_3$ and the unit vector $\mathbf{i_4}$. Hence k_{31} is $k_3\cos(\theta_1)$. The projection of $\mathbf{k}_3$ perpendicular to the $\mathbf{i}_1$ axis is $\mathbf{k}_3\sin(\theta_1)$ and it is a 3-dimensional vector which is resolved in the usual manner. In writing the integral J_1, $\mathbf{k}_2$ is taken along the $\mathbf{i}_1$ axis and the four dimensional volume element

$$d\mathbf{k}_3 = k_3^3\sin^2(\theta_1)\sin(\theta_2)dk_3d\theta_1d\theta_2d\phi,$$

has been used. On multiplying the r.h.s with $\int_0^{\pi}\sin^2(\theta_1)d\theta_1 = \pi/2$, the angular part of the integral can be factored out. Then, with the usual notation, one

gets

$$J_1(k_2) = \frac{2\kappa_4}{\pi}\int\limits_{\Lambda/q}^{\Lambda} k_3 dk_3 \int\limits_0^{\pi} \sin^2(\theta_1) d\theta_1 \Big[k_2^2 + k_3^2 + 2k_2k_3\cos(\theta_1)\Big]^{-1}.$$

The last integral can be evaluated using the result

$$\int\limits_0^{\pi} \sin^2(\theta)[p + q\cos(\theta)]^{-1} d\theta = \frac{p\pi}{q^2}\Big(1 - \sqrt{(p^2 - q^2)/p^2}\Big),\ p > q.$$

The final result is

$$\begin{aligned}
&\int\limits_0^{\pi} \sin^2(\theta_1) d\theta_1 \Big[k_2^2 + k_3^2 + 2k_2k_3\cos(\theta_1)\Big]^{-1} \\
&= \frac{\pi}{4k_2^2k_3^2}[k_2^2 + k_3^2 - \sqrt{(k_2^2 - k_3^2)^2}] \\
&= \frac{\pi}{4k_2^2k_3^2}[k_2^2 + k_3^2 - (k_2^2 - k_3^2)] = \frac{\pi}{2k_2^2},\ k_2 > k_3\ , \\
&= \frac{\pi}{4k_2^2k_3^2}[k_2^2 + k_3^2 - (k_3^2 - k_2^2)] = \frac{\pi}{2k_3^2},\ k_2 < k_3.
\end{aligned}$$

Therefore, $J_1(k_2)$ is found to be

$$\begin{aligned}
J_1(k_2) &= \frac{2\kappa_4}{\pi}\frac{\pi}{2}\int\limits_{\Lambda/q}^{k_2} k_3 \frac{dk_3}{k_2^2} + \int\limits_{k_2}^{\Lambda} k_3 \frac{dk_3}{k_3^2} \\
&= \kappa_4\Big[\frac{k_2^2 - (\Lambda/q)^2}{2k_2^2} + \ln(\frac{\Lambda}{k_2})\Big].
\end{aligned}$$

In a similar way, one gets

$$J_1(\mathbf{k}_1 + \mathbf{k}_2) = \kappa_4\Big[\frac{(\mathbf{k}_1 + \mathbf{k}_2)^2 - (\Lambda/q)^2}{2(\mathbf{k}_1 + \mathbf{k}_2)^2} + \ln(\frac{\Lambda}{|\mathbf{k}_1 + \mathbf{k}_2|})\Big].$$

Therefore, $I_1(k_1)$ becomes

$$\begin{aligned}
I_1(k_1) &= \frac{1}{(2\pi)^4}\frac{1}{c^3}\int\limits_{\Lambda/q}^{\Lambda} d\mathbf{k}_2[J_1(\mathbf{k}_1 + \mathbf{k}_2) - J_1(k_2)] \\
&= \frac{\kappa_4}{(2\pi)^4}\frac{1}{c^3}\int\limits_{\Lambda/q}^{\Lambda} \frac{d\mathbf{k}_2}{k_2^2}\Big[(\frac{\Lambda}{q})^2\frac{1}{2}\{k_2^{-2} - (\mathbf{k}_1 + \mathbf{k}_2)^{-2}\} + \ln(\frac{|\mathbf{k}_2|}{|\mathbf{k}_1 + \mathbf{k}_2|})\Big].
\end{aligned}$$

Now, note that the curly bracket does not contribute for $k_1 < \Lambda/q$. To see this, consider

$$J_1(k_1) = \frac{1}{(2\pi)^4}\int\limits_{\Lambda/q}^{\Lambda} \frac{d\mathbf{k}_2}{k_2^2}(\mathbf{k}_1 + \mathbf{k}_2)^{-2}.$$

For $k_1 < \Lambda/q$, J_1 reduces to

$$J_1(k_1) = \kappa_4 \int_{\Lambda/q}^{\Lambda} dk_2 k_2 \; k_2^{-2} = \frac{1}{(2\pi)^4} \int_{\Lambda/q}^{\Lambda} d\mathbf{k}_2 k_2^{-2} k_2^{-2},$$

which cancels with the first term in the curly bracket in I_1. Hence for small k_1, one gets

$$\begin{aligned} I_1(k_1) &= \frac{\kappa_4}{(2\pi)^4} \frac{1}{c^3} \int_{\Lambda/q}^{\Lambda} \frac{d\mathbf{k}_2}{k_2^2} \ln\Big[\frac{|\mathbf{k}_2|}{|\mathbf{k}_1 + \mathbf{k}_2|}\Big] \\ &= \frac{\kappa_4}{(2\pi)^4} \frac{1}{2c^3} \int_{\Lambda/q}^{\Lambda} \frac{d\mathbf{k}_2}{k_2^2} \ln\Big[\frac{k_2^2}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2}\Big]. \end{aligned}$$

For $k_1 << \Lambda/q$, the logarithm can be expanded as

$$\begin{aligned} &\ln\Big[\frac{k_2^2}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2}\Big] \\ &= \ln(k_2^2) - \ln(k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2) \\ &= -\ln\Big[1 + \frac{k_1^2}{k_2^2} + \frac{2\mathbf{k}_1 \cdot \mathbf{k}_2}{k_2^2})\Big] \\ &= \Big[\frac{k_1^2}{k_2^2} + \frac{2\mathbf{k}_1 \cdot \mathbf{k}_2}{k_2^2} - \frac{4(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{2k_2^4} + O(k_1^3)\Big]. \end{aligned}$$

Thus $I_1(k_1)$ reduces to

$$\begin{aligned} I_1(k_1) &= \frac{\kappa_4}{(2\pi)^4} \frac{1}{2c^3}\Big[- \int_{\Lambda/q}^{\Lambda} d\mathbf{k}_2 \frac{k_1^2}{k_2^4} + 2 \int_{\Lambda/q}^{\Lambda} d\mathbf{k}_2 \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_2^6}\Big] \\ &= \frac{\kappa_4}{2c^3}\Big[- \kappa_4 k_1^2 \int_{\Lambda/q}^{\Lambda} \frac{dk_2}{k_2} + \frac{2k_1^2}{(2\pi)^4} \int_{\Lambda/q}^{\Lambda} \frac{d\mathbf{k}_2}{k_2^4} \cos^2(\theta_1)\Big] \\ &= \frac{\kappa_4}{2c^3} k_1^2\Big[- \kappa_4 \ln(q) + 2\frac{2\kappa_4}{\pi} \int_{\Lambda/q}^{\Lambda} \frac{dk_2}{k_2} \int_0^{\pi} \cos^2(\theta_1) \sin^2(\theta_1) d\theta_1\Big] \\ &= \frac{\kappa_4}{2} \frac{1}{c^3} k_1^2\Big[- \kappa_4 \ln(q) + 2\frac{2\kappa_4}{\pi} \ln(q) \frac{\pi}{8}\Big] \\ &= -\kappa_4^2 k_1^2 \frac{\ln(q)}{4c^3} + O(k_1^3). \end{aligned}$$

Therefore w^* is given by

$$\frac{w^*}{L^{2d}} = \kappa_4^2 \frac{\ln(q)}{4c^3},$$

which yields

$$\eta = \epsilon^2 \Big[\frac{2}{\kappa_4}\frac{c^2}{n+8}\Big]^2 \frac{(n+2)\kappa_4^2}{2c\ln(q)}\frac{\ln(q)}{4c^3} = \frac{\epsilon^2}{2}\frac{n+2}{(n+8)^2} + O(\epsilon^3).$$

Thus, the second order approximation yields a value of η proportional to ϵ^2. However, all the other exponents are accurate to $O(\epsilon)$ only. The reason is that the second order approximation provides the fixed point u^* accurate only to $O(\epsilon)$. The value of r^* could have been obtained to an accuracy of $O(\epsilon^2)$. Further improvements are possible if u^* is calculated more accurately using the third order approximation.

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Chapter 7

Real Space Renormalization Group Methods

The applications of the renormalization group techniques to the Landau-Ginzburg model were discussed in the last two chapters. A systematic perturbation expansion, in the parameter $\epsilon = d - 4$ where d is the spatial dimension, was developed there to calculate the critical exponents to first order accuracy. These results showed that exponents are different from those given by Landau's theory for $d \leq 4$. Higher order calculations are necessary for obtaining accurate values for $d = 3$. In any case, it would be difficult to get good results for two dimensional systems in this manner.

The real space RG techniques are outlined in the present chapter. In these methods, Kadanoff's idea of coarse graining (in real space) is employed in the first step of an RG calculation, i.e., reduction of degrees of freedom. In Chapter 4, this approach was discussed in detail for the 1-D Ising model with nearest neighbour interaction. There, the configurations of every alternate spin, in the definition of the partition function, were summed up to obtain a new system. This system turned out to be again an Ising model with nearest neighbour interactions. However, this is not the case when a similar procedure is attempted in higher dimensions. For example, if one starts with a 2-D Ising model with nearest neighbour interactions and performs a reduction of degrees of freedom by summing the configurations of a set of spins, the resulting system is found to have additional types of interactions. Similar difficulty was seen in the case of the L-G model, there, starting with a quartic term in the hamiltonian, the RG procedure led to higher order terms. Thus, for the RG approach to be useful, either the additional terms generated should be suitably approximated or one should start with a sufficiently general hamiltonian. The generation of additional coupling terms for the Ising model is demonstrated first.

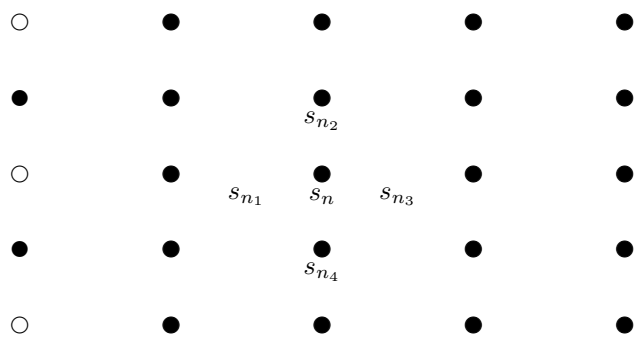

Figure 7.1: Decimation of Square Lattice.

7.1 Need for General Hamiltonians

Consider the Ising hamiltonian for a 2-D square lattice. If there is no external field, the hamiltonian with nearest neighbour $(n.n)$ coupling is

$$\widetilde{H} = -\frac{H}{T} = K_2 \sum_{<i,j>} s_i s_j,$$

where K_2 is the strength of the $n.n$ interaction. The probability distribution of spin configurations is

$$P(\{s_i\}) = \frac{1}{Z} \exp(\widetilde{H}),$$

where the partition function Z is obtained by summing over the configurations of all the spins, that is

$$Z = \sum_{\{s_i\}} \exp(\widetilde{H}).$$

For the 2-D lattice (see Figure 7.1) , the hamiltonian can be rewritten as

$$\widetilde{H} = K_2 \sum_n s_n (s_{n_1} + s_{n_2} + s_{n_3} + s_{n_4}).$$

Here, s_n represents a spin marked with an open circle and s_{n_i} $(1 \leq i \leq 4)$ denotes one of the four surrounding spins. $\sum_n$ stands for summation over the spins indicated by open circles. A new hamiltonian $\widetilde{H}''$ is now obtained by making a partial summation of the configurations of $\{s_n\}$ in the partition function. That is

$$\exp(\widetilde{H}'') = \sum_{\{s_n = \pm 1\}} \exp(\widetilde{H}).$$

Substituting for $\widetilde{H}$, one gets

$$\exp(\widetilde{H}'') \quad = \quad \sum_{\{s_n = \pm 1\}} \exp\left[K_2 \sum_n s_n \{s_{n_1} + \cdots + s_{n_4}\}\right]$$

$$
\begin{aligned}
&= \prod_n \sum_{s_n=\pm 1} \exp\Big[K_2 s_n\{s_{n_1}+\cdots+s_{n_4}\}\Big] \\
&= \prod_n 2\cosh\Big[K_2\{s_{n_1}+\cdots+s_{n_4}\}\Big].
\end{aligned}
$$

The hamiltonian $\widetilde{H}''$ defined above will correspond to a lattice shown in figure 7.1 with filled circles. That is a square lattice rotated by $45^o s$ to the axes of the original lattice, but the spacing is $\sqrt{2}$ times larger. Now, one tries to write the term

$$t_w = 2\cosh\Big[K_2\{s_{n_1}+\cdots+s_{n_4}\}\Big],$$

as an exponential function containing products of s_{n_i} $(1 \le i \le 4)$. Each of the s_{n_i} can take values ± 1. Thus there are $2^4 = 16$ configurations for these spins. t_w can take three values corresponding to

$$s_{n_1}+\cdots+s_{n_4} = \begin{cases} \pm 4 \\ \pm 2 \\ 0 \end{cases}$$

Therefore it can be written as an exponential function containing at least three parameters. More generally, t_w is written as

$$
\begin{aligned}
t_w &= \exp\Big[K_2'(s_{n_1}s_{n_2}+s_{n_2}s_{n_3}+s_{n_3}s_{n_4}+s_{n_4}s_{n_1}) \\
&+ K_2''(s_{n_1}s_{n_3}+s_{n_2}s_{n_4}) + K_4'(s_{n_1}s_{n_2}s_{n_3}s_{n_4}) + c\Big],
\end{aligned}
$$

where K_2' and K_2'' represent the $n.n$ and the next nearest neighbour $(n.n.n)$ coupling constants of the new lattice. Similarly, K_4' denotes the four spin coupling constant. No term containing the product of three spins is present since $\widetilde{H}$ is invariant w.r.t changing all s_i to $-s_i$ and $\widetilde{H}''$ also should have the same symmetry. Putting the values for s_{n_i}, one gets

$$
\begin{aligned}
2\cosh(4K_2) &= \exp(4K_2'+2K_2''+K_4'+c), \\
2\cosh(2K_2) &= \exp(0+0-K_4'+c), \\
2\cosh(0) &= \exp(0-2K_2''+4K_4'+c), \\
2\cosh(0) &= \exp(-4K_2'+2K_2''+K_4'+c).
\end{aligned}
$$

These equations can be solved for the new coupling constants in terms of the old ones. The final results are

$$
\begin{aligned}
K_2' &= \frac{1}{8}\ln[\cosh(4K_2)], \qquad (7.1) \\
K_2'' &= \frac{1}{8}\ln[\cosh(4K_2)], \\
K_4' &= \frac{1}{8}\ln[\cosh(4K_2)] - \frac{1}{2}\ln[\cosh(2K_2)], \\
c &= \frac{1}{8}\ln[\cosh(4K_2)] + \frac{1}{2}\ln[\cosh(2K_2)] + \ln(2).
\end{aligned}
$$

Hence, the new hamiltonian is defined as

$$\begin{aligned}\exp(\widetilde{H}'') &= \exp\Big[\sum_n K_2'(s_{n_1}s_{n_2}+s_{n_2}s_{n_3}+s_{n_3}s_{n_4}+s_{n_4}s_{n_1}) \\ &+ K_2''(s_{n_1}s_{n_3}+s_{n_2}s_{n_4})+K_4'(s_{n_1}s_{n_2}s_{n_3}s_{n_4})+c\Big].\end{aligned}$$

Now, note that when n takes all the values, the $n.n$ coupling between two lattice points comes up two times and there are a total of $N/2$ spins left where N is the total number of original spins. Thus $\widetilde{H}''$ becomes

$$\widetilde{H}'' = 2K_2'\sum s_i s_j + K_2''\sum{}' s_i s_j + K_4'\sum{}'' s_i s_j s_k s_l + \frac{N}{2}c \equiv \widetilde{H}' + \frac{N}{2}c. \quad (7.2)$$

where the $\sum$ is over $n.n$ points, the $\sum'$ is over $n.n.n$ points and the $\sum''$ is over four spins. The definition of the new hamiltonian $\widetilde{H}'$ can be modified as

$$\exp\Big(\widetilde{H}' + \frac{N}{2}c\Big) = \sum_{\{s_n=\pm 1\}} \exp(\widetilde{H}).$$

The probability distribution of the remaining spins is

$$P' = Z'\exp(\widetilde{H}'),$$

where Z', the partition function of the new system, is related to Z as

$$Z'\exp(\frac{N}{2}c) = Z.$$

The definition of free energy per spin F, yields the relation

$$F'[\widetilde{H}'] = 2F[\widetilde{H}] + cT.$$

Note that a different functional form F' has been used for the new system since the new hamiltonian has additional coupling terms. These terms, the second and third summations in Eq.(7.2), arise due to the elimination of degrees of freedom. So one should start with a more general hamiltonian containing such coupling terms (and probably others) before the elimination process so that a closed set of transformation equations are obtained.

As a first step, one may neglect the additional coupling terms and approximate the new hamiltonian with just the first term in Eq.(7.2). In this approximation, $F'[\widetilde{H}'] = F[\widetilde{H}']$ and the transformation is defined solely by Eq.(7.1). This equation is similar to that in the 1-D case and one knows that there is no non-trivial fixed point. A next step could be to neglect the four spin coupling term but suitably approximate the $n.n.n$ interaction term. The energy contributions from the $n.n$ and $n.n.n$ couplings are of same sign and hence they have a tendency to align the spins. For every spin, there are four

$n.n$ and $n.n.n$ bonds. So to include the effect of $n.n.n$ couplings in $\widetilde{H}'$, in a rather crude way, an effective $n.n$ coupling constant is introduced as

$$2K_2' + K_2'' \rightarrow K_2'.$$

This definition conserves the total energy contribution from these types of couplings. Then, Eq.(7.1) can be modified as

$$K_2' = \frac{3}{8}\ln[\cosh(4K_2)]. \tag{7.3}$$

The new hamiltonian is again a $n.n$ Ising model with K_2' as the coupling constant. Eq.(7.3) has fixed points $K_2^* = 0$, $K_2^* = \infty$ and $K_2^* = 0.507$. Linearising it around the non-trivial fixed point, one gets

$$\begin{aligned} \triangle K_2' &= \frac{3}{2}\tanh(4K_2^*)\triangle K_2 \\ &= q^{y_1}\triangle K_2 = (\sqrt{2})^{y_1}\triangle K_2, \end{aligned}$$

where $q = \sqrt{2}$ is the scale factor. Therefore $y_1 \approx 1.07$ and hence the correlation length exponent $\nu = y_1^{-1} \approx 0.935$ which may be compared with the exact value 1 (from Onsager's solution) and the Landau's result 0.5. Thus, even a crude method of implementing the RG procedure can provide a good estimate of the correlation length exponent.

7.2 Spin Decimation - Majority Rule

This method, due to Niemeyer and van Leeuwen, is discussed for the case of Ising model on a triangular lattice. The critical exponents are the same irrespective of the lattice type. As seen in Figure 7.2, the lattice is divided into triangular cells such that each spin is associated with a cell. Let s_i be the i^{th} spin (site spin) of the original lattice, s_i' be the spin (cell spin) associated with the i^{th} cell and s_i^1, s_i^2 and s_i^3 be the three site spins belonging to the i^{th} cell. In the present method, the cell spin s_i' is defined by the 'majority rule'

$$s_i' = sign\ (s_i^1 + s_i^2 + s_i^3).$$

Depending on the values of site spins s_i^α $(\alpha = 1, 2, 3)$, s_i' can take values ± 1. Since the R.G. transformation introduces new types of couplings, one starts with a general hamiltonian

$$\begin{aligned} \widetilde{H}(\{s_i\}) &= -h\sum_i s_i + K\sum_{<i\ j>} s_i s_j + L\sum_{<<i\ j>>} s_i s_j \\ &+ M\sum_{<<<i\ j>>>} s_i s_j + (Four\ Spin\ Coupling) + etc., \end{aligned}$$

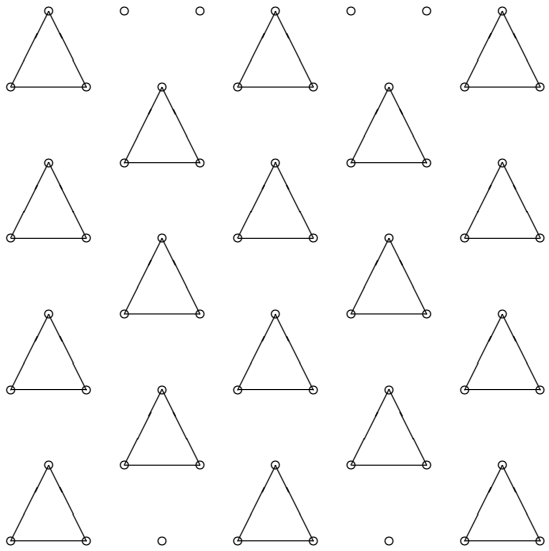

Figure 7.2: Partitioning of Triangular Lattice.

where $K,\ L$ and M are the $n.n,\ n.n.n$ and $n.n.n.n$ coupling constants. For a given value of cell spin s'_i, there are 4 site spin configurations. They are denoted by c_i $(1 \leq i \leq 4)$

$$\begin{aligned}
c_i = 1 \quad &: \quad s_i^1 = \ s'_i, \quad s_i^2 = \ s'_i, \quad s_i^3 = \ s'_i, \\
c_i = 2 \quad &: \quad s_i^1 = -s'_i, \quad s_i^2 = \ s'_i, \quad s_i^3 = \ s'_i, \\
c_i = 3 \quad &: \quad s_i^1 = \ s'_i, \quad s_i^2 = -s'_i, \quad s_i^3 = \ s'_i, \\
c_i = 4 \quad &: \quad s_i^1 = \ s'_i, \quad s_i^2 = \ s'_i, \quad s_i^3 = -s'_i.
\end{aligned}$$

Then $\widetilde{H}(\{s_i\})$ can be expressed as a functional of $\{s'_i\}$ and $\{c_i\}$. That is

$$\widetilde{H}(\{s_i\}) \equiv \widetilde{H}(\{s'_i\}, \{c_i\}).$$

The hamiltonian of the cell system is then defined as

$$\exp\left[\widetilde{H}'(\{s'_i\}) + N'A\right] = \sum_{\{c_i\}} \exp\left[\widetilde{H}(\{s'_i\}, \{c_i\})\right].$$

where N' is the number of cell spins and $N'A$ is the contribution to the free energy from the reduction of degrees of freedom. Once $\widetilde{H}'$ is obtained, the coupling coefficients $\{K', L', M', \cdots\}$ can be determined in terms of $\{K, L, M, \cdots\}$ and the R.G. transformation equations can be obtained. If the $n.n$ distance is unity, the $n.n$ distance of the cell lattice is $\sqrt{3}$. So the spatial rescaling factor $q = \sqrt{3}$. There is no renormalization of the cell spins since they remain ± 1 as in the site lattice. The eigenvalues of the linear transformation are to be then

expressed as $\rho_i = q^{y_i} = (\sqrt{3})^{y_i}$. If there are only two eigenvalues $\rho_1 > 1$ and $\rho_h > 1$, R.G. ideas developed earlier can be applied and the exponents can be computed using the relations

$$\nu = \frac{1}{y_1}, \quad y_h = \frac{1}{2}(d - \eta + 1).$$

For a 2-D lattice, the exact values are $\nu = 1$ and $\eta = 1/4$ and hence $y_1 = 1$ and $y_h = 15/8$.

The hamiltonian $\widetilde{H}$ is split as $\widetilde{H}_0$ and V such that $\widetilde{H}_0$ contains all the coupling terms between spins inside the cells. Then the new hamiltonian can be expressed as

$$\exp[\widetilde{H}'(s') + N'A] = \sum_{\{c_i\}} \exp[\widetilde{H}_0(s', c)] < \exp[V(s', c)] > .$$

The average of any quantity A is defined as

$$< A >= \frac{\sum_{\{c_i\}} A \exp[\widetilde{H}_0(s', c)]}{\sum_{\{c_i\}} \exp[\widetilde{H}_0(s', c)]}.$$

The cumulant expansion of $< \exp(V) >$ is

$$< \exp(V) >= \exp\Big[< V > + \frac{1}{2}(< V^2 > - < V >^2) + \cdots \Big].$$

Therefore one gets

$$\begin{aligned} &\exp[\widetilde{H}'(s') + N'A] \\ &= \sum_{\{c_i\}} \exp[\widetilde{H}_0(s', c)] \exp\Big[< V > + \frac{1}{2}(< V^2 > - < V >^2) + \cdots \Big]. \end{aligned}$$

Now, $\widetilde{H}_0$ which represents the coupling between spins in the cells can be written as

$$\widetilde{H}_0(s', c) = \sum_i \widetilde{H}_{0i}(s', c),$$

where $\widetilde{H}_{0i}$ is the coupling between spins in the i^{th} cell. In each cell, there are only $n.n$ couplings and then

$$\widetilde{H}_{0i}(s', c) = K(s_i^1 s_i^2 + s_i^2 s_i^3 + s_i^3 s_i^1).$$

Noting that

$$s_i^1 s_i^2 + s_i^2 s_i^3 + s_i^3 s_i^1 = \begin{cases} 3 & for \quad c_i = 1 \\ -1 & for \quad c_i = 2, 3, 4 \end{cases}$$

one finds

$$\widetilde{H}_{0i}(s', c) = K(-1 + 4\delta_{c_i\ 1}),$$

and

$$\widetilde{H}_0(s', c) = K \sum_i (-1 + 4\delta_{c_i\ 1}).$$

Thus $\widetilde{H}_0$ has been expressed in terms of $\{c_i\}$. It is independent of $\{s'_i\}$ and this fact simplifies further calculations. Now, note that

$$\begin{aligned}
\sum_{\{c_i\}} \exp[\widetilde{H}_0(s', c)] &= \sum_{\{c_i\}} \exp\Big[K \sum_i (-1 + 4\delta_{c_i\ 1})\Big] \\
&= \prod_i \sum_{c_i=1}^{4} \exp[K(-1 + 4\delta_{c_i\ 1})] \\
&= \prod_i Z_0 = Z_0^N,
\end{aligned}$$

where Z_0 is given by

$$\begin{aligned}
Z_0 &= \exp(3K) + \exp(-K) \\
&+ \exp(-K) + \exp(-K) \\
&= \exp(3K) + 3\exp(-K).
\end{aligned}$$

7.2.1 First Order Approximation

At this order, it is enough to calculate $< V >$. Thus $< \exp(V) >$ is approximated as $\exp(< V >)$. First of all, let there be only $n.n$ coupling. This will illustrate how K is modified to K' by the R.G transformation in the first order approximation. On retaining only the $n.n$ interaction in $\widetilde{H}$, couplings between $n.n$ cells alone contribute to V. The coupling energy between two $n.n$ cells i and j (see Figure 7.3) is then given by

$$V_{ij} = K(s_j^1 s_i^2 + s_j^1 s_i^3).$$

The average of V_{ij} can be calculated as

$$< V_{ij} >= \frac{\sum_{c_i=1}^4 \sum_{c_j=1}^4 \exp[\widetilde{H}_{0i} + \widetilde{H}_{0j}] V_{ij}}{\sum_{c_i=1}^4 \sum_{c_j=1}^4 \exp[\widetilde{H}_{0i} + \widetilde{H}_{0j}]}.$$

The denominator is just Z_0^2. The numerator is

$$\begin{aligned}
Nr. &= K \sum_{c_i=1}^{4} \sum_{c_j=1}^{4} \exp\Big[K(-1 + 4\delta_{c_i\ 1}) + K(-1 + 4\delta_{c_j\ 1})\Big](s_j^1 s_i^2 + s_j^1 s_i^3) \\
&= 2K \sum_{c_j=1}^{4} \exp[K(-1 + 4\delta_{c_j\ 1})] s_j^1 \sum_{c_i=1}^{4} \exp[K(-1 + 4\delta_{c_i\ 1})] s_i^2 \\
&= 2K[\exp(3K) s'_j + \exp(-K) s'_j][\exp(3K) s'_i + \exp(-K) s'_i] \\
&= 2K[\exp(3K) + \exp(-K)]^2 s'_i s'_j.
\end{aligned}$$

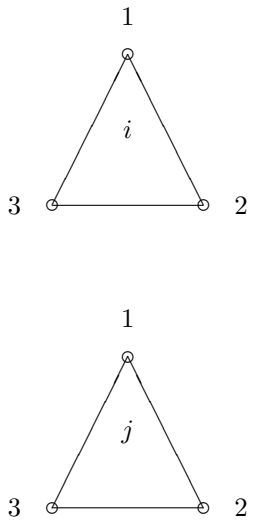

Figure 7.3: Cells i and j.

Hence $< V_{ij} >$ reduces to

$$\begin{aligned} < V_{ij} > &= \frac{2K}{Z_0^2}[\exp(3K) + \exp(-K)]^2 s'_i s'_j = 2K w_1^2 s'_i s'_j, \\ w_1 &= \frac{\exp(3K) + \exp(-K)}{\exp(3K) + 3\exp(-K)}. \end{aligned}$$

Now, considering all the $n.n$ cells, one gets

$$\sum_{<ij>} < V_{ij} >= 2K w_1^2 \sum_{<ij>} s'_i s'_j.$$

Thus, the first order approximation yields

$$\exp[\widetilde{H}'(s') + N'A] = Z_0^N \exp\Big(2K w_1^2 \sum_{<ij>} s'_i s'_j\Big).$$

The new hamiltonian and the free energy term A are given by

$$\begin{aligned} \widetilde{H}'(s') &= 2K w_1^2 \sum_{<ij>} s'_i s'_j \equiv K' \sum_{<ij>} s'_i s'_j, \\ A &= \frac{N}{N'} \ln(Z_0). \end{aligned}$$

It is important to note that $\widetilde{H}'$ contains only the $n.n$ interactions. In higher order approximations, more general coupling terms will appear. The R.G

transformation equation in first order approximation is

$$K' = 2Kw_1^2 = 2K\Big[\frac{\exp(3K)+\exp(-K)}{\exp(3K)+3\exp(-K)}\Big]^2 \equiv f(K).$$

As seen earlier, $K^* = 0$ and $K^* = \infty$ are the trivial fixed points. But the transcendental equation $K^* = f(K^*)$ has a non-trivial fixed point $K^* \approx 0.3356$. Linearising around this fixed point, one gets

$$\Delta K' = \ (\frac{\partial f}{\partial K})_{K^*}\Delta K \quad \approx 1.634\Delta K.$$

Thus the R.G matrix is 1×1 and hence the eigenvalue is $\rho_1 \approx 1.634$. So $y_1 \approx 0.8939$ and $\nu = y_1^{-1} \approx 1.1187$ which may be compared with the exact value 1.

7.2.2 Second Order Approximation

To improve the results of last section, it is necessary to calculate the second term in the cumulant expansion of $< \exp(V) >$. With the $n.n$ interaction between site spins, the term to be evaluated is

$$< V^2 > - < V >^2 = \sum_{<ij>}\sum_{<kl>} < V_{ij}V_{kl} > - < V_{ij} >< V_{kl} > . \qquad (7.4)$$

where $< ij >$ etc. denote $n.n$ cells. If there is no common cell in the pairs $< ij >$ and $< kl >$, then the r.h.s is zero. For $< ij >=< kl >$, r.h.s contributes terms like $< V_{ij}^2 > - < V_{ij} >^2$. The expression for $< V_{ij} >$ obtained in the previous section shows that $< V_{ij} >^2$ is independent of s_i' and s_j'. Following the same steps, one can show that $< V_{ij}^2 >$ also is independent of s_i' and s_j'. Hence such terms contribute only to the free energy term A. So one has to consider only the cases when there is one common cell in the pairs $< ij >$ and $< kl >$. There are four ways to have a common cell j between the pairs $< ij >$ and $< jk >$ and they are shown in Figure 7.4 with numbers 1 to 4 in the respective cells. In Case-1, every pair forms $n.n$ cells. In Case-2 and Case-3, $< ij >$ and $< jk >$ are $n.n$ pairs but $< ik >$ is a $n.n.n$ pair. With $n.n$ interactions, the 3^{rd} spin of j^{th} cell is uncoupled to cells i and k in Case-2. But in Case-3, all the spins of j^{th} cell are connected to other cells. Finally in Case-4, $< ij >$ and $< jk >$ are $n.n$ pairs while $< ik >$ is a $n.n.n.n.$ pair. The contributions to the r.h.s of Eq.(7.4) from the four cases are to be computed.

Case-1 (n.n)

Note that

$$\begin{aligned} < V_{ij}V_{jk} > &= K^2 < (s_i^1 s_j^2 + s_i^1 s_j^3)(s_j^2 s_k^1 + s_j^2 s_k^3) > \\ &= K^2 < s_i^1 s_k^1 + s_i^1 s_k^3 + s_i^1 s_j^3 s_j^2 s_k^1 + s_i^1 s_j^3 s_j^2 s_k^3 > . \end{aligned}$$

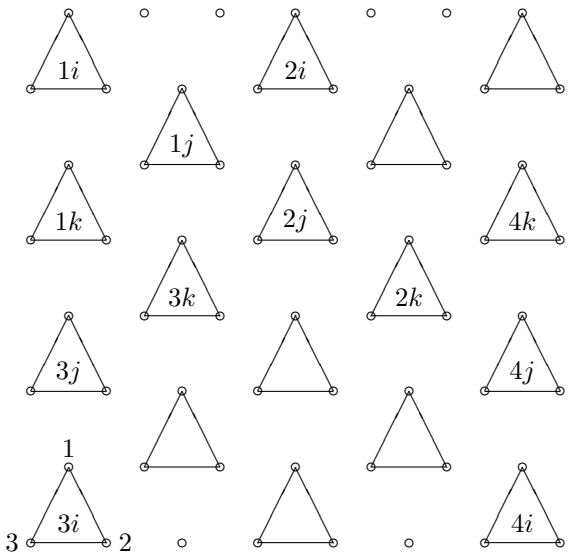

Figure 7.4: Cells for Second Order Calculation.

where the result $[s_j^2]^2 = 1$ has been used. The averages can be easily obtained by following the steps in the calculation of $< V_{ij} >$. Thus

$$\begin{aligned} < s_i^1 s_k^1 > &= < s_i^1 >< s_k^1 > \quad = w_1^2 s_i' s_k', \\ < s_i^1 s_j^3 s_j^2 s_k^1 > &= < s_i^1 s_k^1 >< s_j^3 s_j^2 > \quad = w_1^2 s_i' s_k' w_2, \\ w_1 &= \frac{1}{Z_0}\Big[\exp(3K) + \exp(-K)\Big] \\ w_2 &= \frac{1}{Z_0}\Big[\exp(3K) - \exp(-K)\Big]. \end{aligned}$$

These results lead to

$$< V_{ij} V_{jk} >= K^2(2w_1^2 + 2w_1^2 w_2) s_i' s_k'.$$

The expression already obtained for $< V_{ij} >$ yields

$$< V_{ij} >< V_{jk} >= K^2(2w_1^2 s_i' s_j')(2w_1^2 s_j' s_k') = 4K^2 w_1^4 s_i' s_k'.$$

Thus one gets

$$\frac{1}{2} < V_{ij} V_{jk} > - < V_{ij} >< V_{jk} >= K^2 w_1^2 (a+b) s_i' s_k', \tag{7.5}$$

where a and b are defined as

$$\begin{aligned} a &= 1 - w_1^2 \\ b &= w_2 - w_1^2. \end{aligned}$$

The calculations for the remaining cases follow exactly the same steps and the final results, which can be easily verified, are quoted below.

Case-2 (n.n.n)

$$\frac{1}{2} < V_{ij}V_{jk} > - < V_{ij} >< V_{jk} >= 2K^2 w_1^2 b s_i' s_k'. \tag{7.6}$$

Case-3 (n.n.n)

$$\frac{1}{2} < V_{ij}V_{jk} > - < V_{ij} >< V_{jk} >= K^2 w_1^2 \frac{1}{2}(a+3b) s_i' s_k'. \tag{7.7}$$

Case-4 (n.n.n.n)

$$\frac{1}{2} < V_{ij}V_{jk} > - < V_{ij} >< V_{jk} >= 2K^2 w_1^2 b s_i' s_k'. \tag{7.8}$$

Note that the coupling between i and k cells in Case-1 can be obtained in 4 ways of labelling the cells. First of all, as shown in Figure 7.4, then by the interchange of i and k and then in another two ways by putting the j^{th} cell to the left of i and k. The coupling between i and k in Case-2, Case-3 and Case-4 can be obtained in two ways, first as shown and then by interchanging i and k. Accounting for these extra factors 4 and 2, Eq.(7.5) to Eq.(7.8) can be combined as

$$\begin{aligned} &\frac{1}{2} \sum_{<ij>} \sum_{<jk>} < V_{ij}V_{jk} > - < V_{ij} >< V_{jk} > \\ &= K'' \sum_{<ik>} s_i' s_k' + L'' \sum_{<<ik>>} s_i' s_k' + M'' \sum_{<<<ik>>>} s_i' s_k', \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} K'' &= 4K^2 w_1^2 (a+b), \\ L'' &= K^2 w_1^2 (a+7b), \\ M'' &= 4K^2 w_1^2 b. \end{aligned}$$

Thus it has been shown that $n.n.n$ and $n.n.n.n$ interactions are generated by the partial summation procedure. Therefore, to obtain the transformation laws, the initial site hamiltonian also should have such interactions. It is easy to incorporate these terms and find their contributions to first order accuracy. Referring to Case-1 of Figure 7.4, the contributions from $n.n.n$ and $n.n.n.n$ interactions to V_{ij} can be written as

$$V_{ij}^{(1)} = L(s_i^1 s_j^1 + s_i^2 s_j^2 + s_i^3 s_j^3) + M(s_i^1 s_j^3 + s_i^3 s_j^1),$$

The average of $V_{ij}^{(1)}$ can be easily calculated as

$$< V_{ij}^{(1)} >= (3L + 2M)w_1^2 s'_i s'_j,$$

which provides a $n.n$ contribution

$$\sum_{<ij>} < V_{ij}^{(1)} >= (3L + 2M)w_1^2 \sum_{<ij>} s'_i s'_j.$$

When there is $n.n.n.n$ couplings at the site level, configurations in Case-2 and Case-3 introduce $n.n.n$ coupling between cells i and k. The coupling energy at site level is

$$V_{ik}^{(2)} = \begin{cases} M\ s_i^2 s_k^1 & for\ \ Case-2 \\ M\ s_i^1 s_k^3 & for\ \ Case-3 \end{cases}$$

The average of $V_{ij}^{(2)}$ contributes the $n.n.n$ term

$$\sum_{<<ij>>} < V_{ij}^{(2)} >= Mw_1^2 \sum_{<<ij>>} s'_i s'_j.$$

Adding up the above terms to Eq.(7.9) and including the first order term $< V >$, the second order approximation to $\widetilde{H}'$ is found to be

$$\widetilde{H}'(\{s'_i\}) = K' \sum_{<ik>} s'_i s'_k + L' \sum_{<<ik>>} s'_i s'_k + M' \sum_{<<<ik>>>} s'_i s'_k, \qquad (7.10)$$

where

$$\begin{aligned} K' &= 2w_1^2 K + K'' + (3L + M)w_1^2 \\ &= w_1^2\Big[2K + 4(a + b)K^2 + 3L + 2M\Big] \\ L' &= L'' + w_1^2 M \ = w_1^2\Big[(a + 7b)K^2 + M\Big] \\ M' &= M'' \ = w_1^2 4bK^2. \end{aligned}$$

In summary, the $n.n$, $n.n.n$ and $n.n.n.n$ interactions of site spins have been taken into account to obtain the first order approximation. However, only the $n.n$ interaction is considered in the second order approximation. One may assume that K is much larger than L and M, that is, $K \sim O(1)$ and $L \sim M \sim O(2)$. So the contribution of L and M in the second order approximation is of $O(4)$ and hence can be neglected.

The fixed point values of the transformation equations can be worked out and the results are $K^* \approx 0.2789$, $L^* \approx -0.0143$ and $M^* \approx -0.0152$. These values indicate that the fixed point hamiltonian has ferromagnetic $n.n$ coupling, however, the $n.n.n$ and $n.n.n.n$ couplings are anti-ferromagnetic in nature. The linearised transformation matrix $\widetilde{R}$ is found to be

$$\widetilde{R} \approx \begin{pmatrix} 1.8966 & 1.3446 & 0.8964 \\ -0.0403 & 0.0 & 0.4482 \\ 0.0782 & 0.0 & 0.0 \end{pmatrix}$$

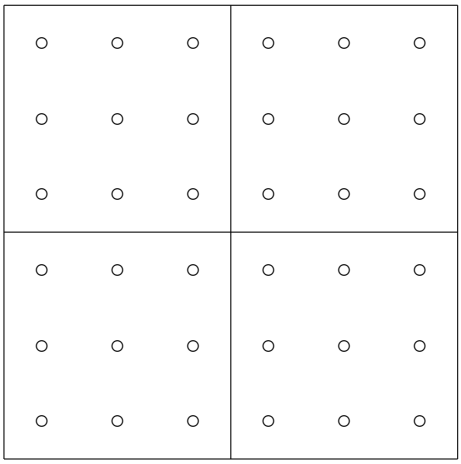

Figure 7.5: Division of Square Lattice in to Cells.

with eigenvalues

$$\begin{aligned} \rho_1 &\approx 1.7835 \\ \rho_2 &\approx 0.2286 \\ \rho_3 &\approx -0.1156. \end{aligned}$$

Thus there is only one eigenvalue $\rho_1 > 1$ and hence the assumptions in the R.G theory are valid. Now, $\rho_1 \approx 1.7835 = (\sqrt{3})^{y_1}$ yields $\nu = 1/y_1 \approx 0.9494$. This value is better than that obtained in the first order approximation, however, the convergence to the exact value is slow.

7.2.3 Square Lattice

The procedure outlined above can also be applied for the case of a square lattice. To use the majority rule to define the cell spin, it is necessary that each cell contains an odd number of site spins. Thus there has to be a minimum of nine spins per cell (see Figure 7.5). The cell lattice is also square but with a spacing three times that of the site lattice. The majority rule definition of cell spin is

$$s'_i = sign\ (s_i^1 + s_i^2 + \cdots + s_i^9),$$

and s'_i takes values ± 1.

7.3 Formulation using Weight Functions

The schemes discussed above can be formulated in a general way using a weight function $P(\{s'_i\},\{s_i^\alpha\})$ which depends on N' cell spins $\{s'_i\}$ and N site spins $\{s_i^\alpha\}$. For the triangular lattice, P is defined as

$$P(s',s) \equiv \prod_i p(s'_i,\{s_i^\alpha\}),$$

where $p(s'_i,\{s_i^\alpha\})$ has the form

$$p(s'_i,\{s_i^\alpha\}) = \prod_i \frac{1}{2}\Big[1+\frac{1}{2}s'_i\{(s_i^1+s_i^2+s_i^3)-s_i^1s_i^2s_i^3\}\Big]. \tag{7.11}$$

Note that the factor $p(s'_i,\{s_i^\alpha\})$ becomes unity when

$$s'_i = sign\ (s_i^1+s_i^2+s_i^3),$$

and 0 otherwise. The renormalized hamiltonian $\widetilde{H}'(\{s'_i\})$ is defined as

$$\exp[\widetilde{H}'(\{s'_i\})] = \sum_{\{s_i^\alpha=\pm1\}} P(s',s)\exp[\widetilde{H}(s)],$$

where the sum is over the configurations of site spins. The sum contributes only when the majority rule is satisfied and so the cell spins $\{s'_i\}$ take values ±1 just like the site spins. A particular configuration of site spins corresponds to a specific value of energy, $\widetilde{H}(s)$. Corresponding to that configuration, there is a configuration of cell spins determined by the weight function $P(s',s)$. Then, summing over all the configurations of site spins yields the cell energy function $\widetilde{H}'(s')$. This formulation of reducing the degrees of freedom is very general, however, some minor restrictions are required on P. First of all, one must have

$$P(s',s) \geq 0,$$

for any configuration of $\{s_i^\alpha\}$ and $\{s'_i\}$ since it assigns weights to configurations of site spins. Secondly, the partition function $Z_{N'}$ of the cell model,

$$\begin{aligned} Z_{N'} &= \sum_{\{s'_i\}} \exp(\widetilde{H}') \\ &= \sum_{\{s'_i\}}\sum_{\{s_i^\alpha\}} P(s',s)\exp(\widetilde{H}), \end{aligned}$$

should be the same as that of the site model. So, there is a restriction,

$$\sum_{\{s'_i\}} P(s',s) = 1$$

for every configuration of the site spins. The weight function of Eq.(7.11) can be generalized as

$$p(s'_i, \{s_i^\alpha\}) = \prod_i \frac{1}{2}\Big[1 + s'_i f(s_i^1, s_i^2, \cdots s_i^n)\Big],$$

where s_i^α, $1 \leq \alpha \leq n$ are the spins in the i^{th} cell and f is some suitable function.

The method of decimation introduced earlier (section 7.1) for the square lattice can also be interpreted in terms of a weight function. There, spins on alternate diagonals were decimated to obtain another square lattice with a spacing $\sqrt{2}$ times larger. If the spins not to be decimated are indicated as s'_i and those to be decimated as s_i, then the definition of the new hamiltonian $\widetilde{H}'(\{s'_i\})$ is

$$\exp[\widetilde{H}'(\{s'_i\})] = \sum_{\{s_i\}} \exp[\widetilde{H}(\{s'_i\}, \{s_i\})].$$

Using the identity

$$\sum_{s_j} \frac{1}{2}(1 + s'_i s_j) f(s_j) = \begin{cases} f(1) & if \;\; s'_i = 1 \\ f(-1) & if \;\; s'_i = -1 \end{cases} \equiv f(s'_i).$$

for any function $f(s_i)$, one can write

$$\exp[\widetilde{H}(\{s'_i\}, \{s_i\})] = \sum_{\{s_j\}} \Big[\prod_j \frac{1}{2}(1 + s'_i s_j)\Big] \exp[\widetilde{H}(\{s_j\}, \{s_i\})].$$

Then, summation over the configurations of s_i yields

$$\exp[\widetilde{H}'(\{s'_i\}] = \sum_{\{s_i\}} \sum_{\{s_j\}} \Big[\prod_j \frac{1}{2}(1 + s'_i s_j)\Big] \exp[\widetilde{H}(\{s_j\}, \{s_i\})].$$

This is then identical to the general definition with the weight function

$$P(s', s) = \prod_j \frac{1}{2}(1 + s'_i s_j).$$

The formulation of reduction of degrees of freedom in terms of weight function has the flexibility that any appropriate form for the same can be employed. Kadanoff's proposal of a weight function for the d-dimensional cubic lattice is discussed in the following section.

7.4 Kadanoff's Bond Moving Technique

Kadanoff's method to derive approximate RG transformation equations is based on the idea that the hamiltonian of the model, $\widetilde{H}(s)$, can be replaced

by $\widetilde{H}(s) + V(s)$ where $V(s)$ is to be chosen appropriately. The renormalized hamiltonian $\widetilde{H}'(s)$ is defined as

$$\exp[\widetilde{H}'(s')] = \sum_s P(s', s) \exp[\widetilde{H}(s) + V(s)].$$

Since the original hamiltonian has been modified, the free energy calculated with $\widetilde{H}'(s)$ will not be the same as that with $\widetilde{H}(s)$. However, by choosing $V(s)$ appropriately, the free energy corresponding to $\widetilde{H}'(s)$ can be made a lower bound to the original free energy. Using the inequality, $\exp(x) \geq 1 + x$, and summing over the cell spin configurations, one gets

$$\sum_{s'} \exp[\widetilde{H}'(s')] \geq \sum_s \exp[\widetilde{H}(s)][1 + V(s)],$$

where the normalization condition on the weight function $P(s', s)$ has been used. This relation implies that

$$Z(\widetilde{H}') \geq Z(\widetilde{H})[1 + < V(s) >],$$

where $Z(\widetilde{H}')$ and $Z(\widetilde{H})$ are respectively the partition functions of the reduced and original model and $< V(s) >$ is the statistical average of $V(s)$ with the Boltzmann weight $\exp[\widetilde{H}(s)]$. Then, the condition

$$< V(s) >= 0,$$

yields the inequality

$$F_t(\widetilde{H}') \leq F_t(\widetilde{H}),$$

since the free energy $F_t = -T \ln(Z)$. Thus the free energy of the system resulting out of the transformation is a lower bound to the exact free energy. This observation leads to the possibility of introducing some variational parameters in $P(s', s)$ so that $F_t(\widetilde{H}')$ can be maximized w.r.t those parameters and a good approximation to the exact free energy can be obtained. As an example of choosing $V(s)$ for the $2 - D$ Ising model hamiltonian $\widetilde{H}(s)$, consider

$$V(s) = K(s_3 s_4 - s_1 s_2).$$

Then it is clear that $\widetilde{H}(s) + V(s)$ represents a system in which the coupling term between the spins s_1 and s_2 is absent but that between s_3 and s_4 has double its original value. This particular choice has the required property, $< V(s) >= 0$, due to the translation symmetry of the (infinite) lattice. This method of constructing the interaction potential $V(s)$ by spatially shifting the coupling energies between spins is called the bond moving technique. The method, thus, shifts the 'troublesome' coupling terms in $\widetilde{H}(s)$ so that the RG transformation with $\widetilde{H}(s) + V(s)$ can be easily effected.

Table 7.1: Critical Exponents - Bond Moving Technique

d	ν	ν_{exact}	δ	δ_{exact}
2	0.999	1.000	15.04	15.00
3	0.619	0.630	4.604	4.82
4	0.491	0.500	2.9	3.00

In Kadanoff's calculations for the d-dimensional Ising model, the lattice is divided into cells of length twice the original lattice spacing. If the spins belonging to the i^{th} cell are denoted by $s_i^\alpha,\ 1 \leq \alpha \leq 2d$, the weight function is defined as

$$P(s', s) = \prod_i p(s_i', \{s_i^\alpha\}),\ 1 \leq \alpha \leq 2d,$$

$$p(s_i', \{s_i^\alpha\}) = \frac{1}{2} \frac{\exp[\lambda s_i'(s_i^1 + \cdots + s_i^{2d})]}{\cosh[\lambda(s_i^1 + \cdots + s_i^{2d})]},$$

where λ is a parameter to be adjusted. By adding a suitable term $V(s)$ representing moving of bonds, Kadanoff derived approximate transformation equations for the coupling constants. These equations and hence their fixed points depend on the parameter λ. The prescription used to fix the value of λ is that the free energy corresponding to the fixed point hamiltonian is a maximum. This method provides excellent values of critical exponents as seen in Table 7.1.

7.5 Migdal - Kadanoff Method

The Migdal-Kadanoff method is another approximate technique to carry out the reduction of degrees of freedom. Here the spin decimation method together with the bond moving idea is used to develop the transformation equations for multidimensional lattices. In effect, it leads to a generalization of the transformation equations of the $1-D$ lattice. The derivation discussed below is due to Kadanoff who has developed a reinterpretation of Migdal's results using bond moving technique. In Chapter 4, the transformation equations for the $1-D$ lattice were derived by decimating every alternate spins. Thus, with a scale factor $q = 2$, the equations obtained were

$$\exp(4K') = \frac{\cosh(2K + h)\cosh(2K - h)}{\cosh^2(h)},$$

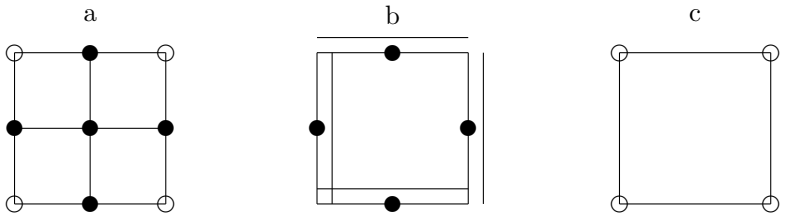

Figure 7.6: Bond-Moving and Decimation.

$$\exp(2h') = \frac{\cosh(2K+h)}{\cosh(2K-h)}\exp(2h).$$

The coupling energy represented by each bond in the $1-D$ lattice is proportional to K. The magnetic field coupling energy per site, which is proportional to h, can also be associated with each bond of the lattice. After the decimation, the resulting lattice will have coupling and magnetic field energies proportional to K' and h' respectively. The fixed point value of h is 0 as seen from the second equation. Since one is interested in the transformation equations near the fixed point, they can be simplified by assuming that h is small. A Taylor expansion around $h = 0$ yields

$$\begin{aligned} K' &= \frac{1}{2}\ln[\cosh(2K)] + O(h^2), \\ h' &= h[1+\tanh(2K)] + O(h^3). \end{aligned}$$

For generalizing these results to the $2-D$ lattice, the bond moving technique is used. The spins to be decimated in the original $2-D$ lattice (Figure 7.6 a) are marked with filled circles. The resulting lattice (Figure 7.6 c) will then have twice the original spacing. Before performing the decimation, an approximate lattice (Figure 7.6 b) is generated by moving certain bonds. All the four bonds within the big square have been moved to the left and bottom. The spins to be decimated are again denoted by filled circles. It is clear that the hamiltonian of the modified lattice can be expressed as $\widetilde{H}(s) + V(s)$ with the property $< V(s) >= 0$ and hence the free energy of the decimated lattice is a lower bound to the exact free energy. Since the spins to be decimated in the modified lattice are coupled only along one direction, the RG transformation can be done exactly as in the $1-D$ case. Each of the bonds in the approximate lattice has twice the original coupling energy. Therefore, to obtain the coupling constant K' of the decimated lattice, one should replace

K by $2K$ in the transformation equations for $d = 1$. Similarly, due to moving of bonds, each bond in the modified lattice carries a field energy proportional to $2h$. Therefore, the magnetic field h' of the decimated lattice is obtained by replacing h by $2h$ in the $1 - D$ equation. Thus for the $2 - D$ lattice, K' and h' are given by

$$\begin{aligned} K' &= \frac{1}{2}\ln[\cosh(4K)] + O(h^2), \\ h' &= 2h[1 + \tanh(4K)] + O(h^3). \end{aligned}$$

The fixed point value of h is $h^* = 0$. The equation for K' has a non-trivial fixed point $K^* \approx 0.305$. Note that there were only the trivial fixed points $K^* = 0$ and $K^* = \infty$ in the $1 - D$ case. Linearising the equations around K^* and h^* one gets

$$\begin{aligned} \Delta K' &= 2\tanh(4K^*)\Delta K, \\ \Delta h' &= 2[1 + \tanh(4K^*)]\Delta h. \end{aligned}$$

These are already in the diagonal form and hence the eigenvalues of the RG matrix are

$$\begin{aligned} \rho_1 &= 2\tanh(4K^*) = 2^{y_1}, \\ \rho_h &= 2[1 + \tanh(4K^*)] = 2^{y_h}, \end{aligned}$$

which yield $y_1 \approx 0.748$ and $y_h \approx 1.879$. These results are to be compared to the exact values 1 and 15/8 respectively. The correlation function exponent $\nu = y_1^{-1} \approx 1.34$ is somewhat inaccurate. The critical coupling constant $K^* \approx 0.305$ (which is related to the critical temperature T_c) compares better with the exact value of 0.44069 obtained from the Onsager solution.

The transformation equations can be easily generalized to the case of a d-dimensional lattice. For that, imagine a big 'cube' of side length twice the original lattice spacing. All the spins except those at the corners of this cube are to be decimated to produce a lattice of double the original spacing. There are a total of 2^d small cubes within the big cube. With each of these small cubes, one can associate d bonds (i.e. those along the d axes). Thus $d\ 2^d$ bonds are associated with the big cube. Out of these, $d\ 2^d - 2d$ bonds have to be moved towards the $2d$ bonds along the d axes of the big cube. The $2d$ bonds are those which are to be decimated in the $1 - D$ procedure. So the bond strength of the modified lattice is $1 + (d\ 2^d - 2d)/2d = 2^{d-1}$. There is a field energy proportional to h with each of the bonds in the original lattice. So the modified lattice field energy is higher by a factor 2^{d-1}. Therefore K' and h' of the decimated lattice are obtained by replacing K and h in the $1-D$

Table 7.2: Exponents - Migdal - Kadanoff Method

d	K^*	ν	δ
2	0.305	1.34	15.5
3	0.0653	1.06	5.90
4	0.0158	1.01	4.87

equations by $2^{d-1}K$ and $2^{d-1}h$ respectively. That is

$$\begin{aligned} K' &= \frac{1}{2}\ln[\cosh(2^d K)] + O(h^2), \\ h' &= 2^d h[1 + \tanh(2^d K)] + O(h^3). \end{aligned}$$

As found earlier, $h^* = 0$. The values of K^* and the exponents ν and δ shown in Table 7.2 are easily computed. The values of the correlation length exponent ν are very inaccurate. When the bonds are shifted to effect the reduction of degrees of freedom, the spatial correlations are not accounted properly and hence the values of ν turn out to be quite wrong. However, the dependence of the exponent δ on d comes out somewhat reasonably in spite of the crude approximations involved.

7.6 Monte Carlo Renormalization

The Monte Carlo method in statistical physics is a powerful tool for calculating the thermodynamic properties of systems away from critical points. Most of the applications of the method have been for discrete models and so the Ising model is considered for illustrating the ideas. Usually one starts with a cubic lattice of linear size L and assumes periodic boundary conditions for the spin orientations at the surfaces. An initial spin configuration c_0, which is rather arbitrary, is first selected by assuming that all spins take value 1. A sequence of configurations $\{c_n\}$ is then generated by successively flipping the spins one by one. A Monte Carlo step (MCP) is said to be over when the spins at every site have been flipped once. The process is then repeated by completing the 2^{nd}, 3^{rd}, etc. MCS. When the system is in equilibrium at temperature T, the probability of occurrence of the n^{th} configuration is

$$P_n = Z^{-1}\exp(-E_n/k_B T),$$

where E_n is the energy associated with that configuration. E_n is readily obtained if the coupling constants K and h are given specific values. The statistical average of any quantity which depends on the spin variables can then

be calculated by weighting its contribution from configuration c_n with P_n. It is a usual practice to consider the configurations at the end of every MCS for calculating the averages so that they are somewhat independent. However, since the main aim is to evaluate the equilibrium properties, it is necessary that the configurations chosen are in the neighbourhood of the most probable one. A direct method of choosing the sequence $\{c_n\}$ such that one approaches the most probable one is to accept only those c_ns for which $E_n < E_{n-1}$. Such a procedure has two drawbacks, (i) a large number of configurations generated will get rejected, and (ii) there is the possibility of the process getting trapped in a local energy minimum. So in the commonly used method, due to Metropolis, the configuration c_n is not rejected always even if $E_n > E_{n-1}$. The basic idea of the method becomes clear if one associates a discrete Markov process with the process of generating configurations. Then c_n represents a state of the process and P_n is the probability of realizing the n^{th} state. The specification of the Markov process becomes complete if the probability of transition from state n to m, $W(n \to m)$, is also defined. The the process of generating $\{c_n\}$, which is analogous to the time development of a system, should eventually lead to equilibrium configurations. For guaranteeing this requirement, it is sufficient that $W(n \to m)$ satisfy the detailed balance condition

$$P_n W(n \to m) = P_m W(m \to n),$$

which means that the probability of realizing the states m and n in succession is equal to that for the reverse event. Such a condition ensures that the probability of occurrence of state n is P_n when the initial transients have died out. Now, if $E_m \leq E_n$, then c_n should be definitely accepted and so $W(n \to m)$ must be unity. Therefore, $W(m \to n)$ reduces to $\exp(-\Delta E/K_B T)$ where $\Delta E = E_n - E_m > 0$. In the Metropolis method, the configuration c_n resulting from c_m with $E_n > E_m$ is accepted with probability $p = \exp(-\Delta E/k_B T)$. The probability that a random number r, uniformly distributed in [0,1], has value less than p is p itself. So, in the process of generating configurations, if E_n turns out to be greater than E_{n-1}, a random number r is computed, and if $r < p$, c_n is accepted, otherwise it is rejected. Approach to equilibrium can be monitored by examining the stabilization of average values. The results obtained with this approach have to be repeated for different values of the system size L and finally extrapolated to the infinite system limit. The inherent statistical errors in the results are well characterized and so very reliable results can be obtained though computer resources usually set the limits of achievable accuracy.

When the system is close to the critical point, the method runs into problems because of two reasons. First of all, the inherent critical slowing down makes the process of approaching equilibrium very difficult. Secondly, the cor-

relation length becomes larger than the size of the simulated system and thus the singularities of thermodynamic quantities at the critical point get rounded off. A way to circumvent this problem is to use the finite size scaling forms for the thermodynamic quantities and extract the critical exponents and other universal features. However, a combination of the Monte Carlo method and real space renormalization group techniques has provided excellent results in the critical region. The simplest idea is to simulate a system of size $L \times L \times 2L$ (in 3 dimension), and consider it as just two cells of size $L \times L \times L$ each. So the cell size parameter q is L. For every site spin configuration, the cell spins s'_1 and s'_2 can be readily obtained by the majority rule. Now, assuming an Ising hamiltonian for the cell system, the new coupling constants K' and h' are extracted from the values of the averages $< s'_1 s'_2 >$ and $< s'_1 >$. Then by changing K and h, the RG fixed point and the matrix elements $\partial K'/\partial K$, $\partial K'/\partial h$, etc. can be numerically computed, and the eigenvalues and the exponents y_1 and y_h determined. The calculations are then repeated for different values of L and extrapolated to the infinite lattice limit.

It is also possible to incorporate other types of couplings (say, the $n.n.n$ coupling) between the spins. A useful procedure has been to adopt the ideas of real space renormalization group techniques more directly. Thus, one starts with a finite lattice of size $L \times L \times L$, generates a sufficient number of equilibrium configurations and stores them in the computer memory. This forms the first set of configurations $\{c_0\}$. Corresponding to every one of these site spin configuration, a cell spin configuration is obtained by majority rule. Thus a set $\{c'_n\}$ is generated. Here, a cell may contain only $2^3 = 8$ site spins corresponding to a cell size $q = 2$. At this stage, the size of the system is $L/2 \times L/2 \times L/2$. Let the hamiltonian of the original system be expressed as

$$\frac{H}{T} = \sum_{\alpha} K_\alpha s_\alpha,$$

where s_α represents the contribution from the interactions of type α which can be $n.n$, $n.n.n$, four spin coupling, magnetic field coupling, etc. and K_α is the corresponding coupling constant. The hamiltonian characterizing the set of configurations $\{c'_n\}$ is then given by

$$\frac{H'}{T} = \sum_{\alpha} K'_\alpha s'_\alpha.$$

The averages $< s_\alpha >$ and $< s'_\alpha >$ can be readily computed and so one can write

$$\frac{\partial}{\partial K_\beta} < s'_\gamma > = \sum_{\alpha} \frac{\partial}{\partial K'_\alpha} < s'_\gamma > \frac{\partial K'_\alpha}{\partial K_\beta}.$$

Now, $\{\partial K'_\alpha/\partial K_\beta\}$ is the RG matrix. The derivatives of the averages can be expressed in terms of their correlations which are calculated easily. The

equivalence of the averages corresponding to H and H' implies that

$$< s'_\gamma >= \sum_{\{c'_n\}} s'_\gamma Z^{-1} \exp(-H'/T) = \sum_{\{c_n\}} s'_\gamma Z^{-1} \exp(-H/T),$$

where the partition function $Z = Z(\{K_\alpha\}) = Z(\{K'_\alpha\})$. Therefore one finds

$$\begin{aligned} \frac{\partial}{\partial K_\beta} < s'_\gamma > &= < s'_\gamma s_\beta > - < s'_\gamma >< s_\beta >, \\ \frac{\partial}{\partial K'_\alpha} < s'_\gamma > &= < s'_\gamma s'_\alpha > - < s'_\gamma >< s'_\alpha > . \end{aligned}$$

So the RG matrix can be numerically computed by solving a set of linear equations. The whole procedure can now be repeated by performing a second level of coarse graining to obtain $\{c''_n\}$. Practical calculations including as large as ten types of coupling constants can be performed quite easily with this approach. These methods have been applied successfully to problems in percolation theory, polymer physics, etc. to be discussed in the next chapter.

7.7 Application to First Order Transitions

First order transitions are accompanied by a discontinuous change in the order parameter across the transition point. The magnetic transition across the temperature axis below T_c, the liquid-gas transition (i.e. boiling) across the vapour pressure curve for $T < T_c$, solid to liquid transition (i.e. melting) across the melting curve, etc. are common examples. In the last two cases, the latent heat accompanying the transition indicates a discontinuous change in the internal energy also. For the magnetic example, the order parameter in the two 'phases' of the system coexisting at $h = 0$ are magnetization in opposite directions. Both phases have only finite correlation lengths and so there are no universal features like scaling of thermodynamic variables. Nevertheless, it is of interest to see if the discontinuity in the order parameter can be associated to an RG fixed point. Repeated coarse graining of the system on either side of the temperature axis (or phase boundary) would remove all fluctuations in spin orientations as the correlation length is finite. Eventually, the RG iterations will produce a system with magnetization in a specific direction and further iterations would leave the system unaltered. This asymptotic state would also be independent of the short length scale details of the original system. Thus it appears that repeated application of the RG transformation drives the system to a fixed point μ^{**} for $T < T_c$. This fixed point must be different from μ^* associated with the critical point. The unidirectional orientation of the spins in the renormalized system represented by μ^{**} is analogous to that occurring at $T = 0$. In fact, the presence of such a fixed point is evident

from the approximate transformation equations derived in the earlier sections. For instance, the recursion relations of the Migdal-Kadanoff method shows that $K^{**} = \infty$ and $h^{**} = 0$ is a fixed point. One can easily see that the eigenvalue exponents y_1 and y_h corresponding to this fixed point are $d-1$ and d respectively, where d is the spatial dimension. The value $K^{**} = \infty$ indicates the strong coupling limit which corresponds to the state at $T = 0$. The result $y_h = d$ is a general property of μ^{**} if it is to represent a discontinuous transition. To show this, consider the functional equation for free energy density,

$$F(h, \bar{\mu}) = q^{-d} F(h', \bar{\mu}') + A(q, \bar{\mu}),$$

where $\bar{\mu}$ denotes the parameter space corresponding to the even part of the hamiltonian. Using the transformation law $h' = q^{y_h} h$ and the relation $m = -\partial F/\partial h$, the discontinuity,

$$\Delta m(\bar{\mu}) = m(0_+, \bar{\mu}) - m(0_-, \bar{\mu}),$$

in the magnetization at $h = 0$can be expressed as

$$\Delta m(\bar{\mu}) = q^{-d+y_h} \Delta m(\bar{\mu}').$$

The term A which must be analytic in h does not contribute to $\Delta m(\bar{\mu})$. Putting $\bar{\mu} = \bar{\mu}' = \bar{\mu}^{**}$, one finds that the field exponent $y_h = d$ provided $\Delta m(\bar{\mu}^{**}) \neq 0$. Now, on repeatedly applying the transformation l times, $\Delta m(\bar{\mu})$ can be written as

$$\Delta m(\bar{\mu}) = q^{l(-d+y_h)} \Delta m(\bar{\mu}^l) \approx q^{l(-d+y_h)} \Delta m(\bar{\mu}^{**}) \ , \ l \gg 1,$$

since $\bar{\mu}^l$ approaches $\bar{\mu}^{**}$ for large l. Thus, if $y_h = d$ and $\Delta m(\bar{\mu}^{**}) \neq 0$, then $\Delta m(\bar{\mu}) \neq 0$. Therefore the fixed point μ^{**}, usually called the discontinuity fixed point, to be associated with a first order transition must have the property $y_h = d$. The transformation equations obtained using the majority rule for a $2-D$ triangular lattice also show this property. There it was shown that the first order approximation to the $n.n$ coupling constant is $K' = 2Kw_1^2(K)$ where the function $w_1(K)$ tends to unity as $K \to \infty$. It is easy to add a magnetic field term to the hamiltonian and then derive the equation $h' = 3hw_1(K)$ for small h. These equations clearly show that $K^{**} = \infty$ and $h^{**} = 0$ is a fixed point and the exponents are (using $q = \sqrt{3}$) $y_1 \approx 1.262$ and $y_h = 2 = d$ respectively.

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Chapter 8

Problems with Many Length Scales

The renormalization group theory of second order (thermal) phase transitions was introduced and developed in the preceding chapters. Systems near the critical point have spatial structures at all length scales and in the RG method, the details of the structure are averaged out in a recursive manner. At each step, the method generates a coarser description of the system in terms of renormalized model parameters. Due to the presence of an infinite length scale, the coarse graining procedure leaves the system unaltered at the critical point. The universal aspects of critical behaviour are related to the variation of the model parameters in the vicinity of the fixed point. All these ideas have been discussed earlier with reference to the simple ferromagnetic critical point. The critical behaviour observed in the condensation of a gas or in a binary mixture can also be mapped into the Ising model of ferromagnetism. Thus there is already a rich area of application of the RG idea, however, its applicability is much wider. In this chapter, a few more problems where the idea has turned out to be of great use are discussed. Just as in the magnetic case, the emphasis is on the universal aspects observed in these phenomena. More details of these applications can be obtained from the cited references.

8.1 Critical Dynamics

The dynamical aspects of critical phenomena form one of the important areas where the RG ideas are very fruitful. In dynamics, one is concerned with the time evolution of a system perturbed from a thermodynamic equilibrium state. A slight change in the macroscopic conditions such as temperature or the magnetic field can trigger the time evolution of the system. The detailed time variation of all the system variables affects the relaxation of the system

to a new equilibrium state. However, one can observe a typical macroscopic time interval, called the relaxation time, in the evolution of the system. The relaxation time depends on the magnitude of the perturbation and, in general, is different at different spatial locations if the perturbation is non-uniform. The dynamical evolution can be considered at various levels of details, for instance, at the level of Ising spins, at a some what coarser level in terms of the spin density in the Landau - Ginzburg model or even at a macroscopic level employing the over all magnetization. However, the present discussion is in terms of the L-G model spin density $s(\mathbf{x})$ or its equivalent Fourier components $\{s_{\mathbf{k}}\}$. Near the critical point, $s(\mathbf{x})$ should change in large regions of size of the correlation length and so dynamics is rather slow. This feature of relaxation process near the critical point is termed as critical slowing down. In fact at $T = T_c$, the relaxation time diverges and the characteristic exponent describing the divergence is one of the important quantities obtained from studies in critical dynamics. To formulate the time evolution of the Fourier amplitudes $s_{\mathbf{k}}$, it is necessary to set up equations of motion for them. For simplicity, consider the quadratic approximation to the one component (n=1) L-G hamiltonian for $T > T_c$,

$$H(\{s_{\mathbf{k}}\}) = \sum_{k \leq \Lambda} (a_2 + ck^2)|s_{\mathbf{k}}|^2.$$

The hamiltonian can be taken as a potential energy function in the variables $\{s_{\mathbf{k}}\}$. Then the negative gradient of H w.r.t $s_{\mathbf{k}}$ is a thermodynamic force driving its time evolution. That is

$$F(\ thermodynamic\ ) = -\frac{\partial}{\partial s_{-\mathbf{k}}} H.$$

It should be noted that the Fourier variables $s_{\mathbf{k}}$ and $s_{-\mathbf{k}}$ are not independent, in fact, they are just complex conjugates. The real and imaginary parts of $s_{\mathbf{k}}$ are independent and hence the force along the co-ordinate $s_{\mathbf{k}}$ should be obtained by calculating the derivatives of H (which also can be expressed in terms of the real and imaginary parts of $s_{\mathbf{k}}$) w.r.t these independent parts and then suitably adding them up. The resulting expression can also be expressed as the derivative of H w.r.t $s_{-\mathbf{k}}$ as shown in the above equation.

The dynamical variables which have been averaged out in the coarse description employing $s(\mathbf{x})$ also affect its time evolution. Further, the coupling of the spin variables with other degrees of freedom (which is one of the reason necessitating a statistical treatment) also influence the dynamics of $s(\mathbf{x})$. Both these contributions vary at a much faster time scale and so they appear as random influences on the spin density. These forces may be separated into a macroscopic frictional force and a purely random force with zero mean value.

These effects manifest in the Brownian motion of suspended particles in a fluid medium. Following the theory of Brownian motion, the friction force, which introduces energy dissipation and irreversibility, can be represented as

$$F(\ friction\ ) = -\Gamma'_{\mathbf{k}} \frac{d}{dt} s_{\mathbf{k}},$$

where $\Gamma'_{\mathbf{k}}$ is a phenomenological friction constant. The random part of the influence of the fast varying degrees of freedom is denoted as

$$F(\ random\ ) = \zeta(t).$$

The function $\zeta(t)$ is the sum of the influences of a very large number of variables and using the central limit theorem of statistics, its distribution can be assumed to be Gaussian. Another reasonable assumption regarding $\zeta(t)$ is that it is uncorrelated at different times. Since it represents the fast varying influences, its values at different times (in the time scale of $s_{\mathbf{k}}$) are assumed to be independent. Thus the correlation function $< \zeta(t)\zeta(t') >$ is

$$< \zeta(t)\zeta(t') >= 2k_B T \delta(t - t'),$$

where the amplitude factor is chosen to be $2k_B T$ so that the equilibrium probability distribution of $s_{\mathbf{k}}$ is proportional to the Boltzmann factor.

In studying the relaxation of the Fourier amplitudes $s_{\mathbf{k}}$ near T_c , one is interested only in their slow time variation. On such a time scale, their second order time variations may be neglected. The time variation of $s_{\mathbf{k}}$ is then obtained by balancing the different force terms. The resulting equation of motion is similar to that of a macro-particle moving with the terminal velocity in a viscous medium. Thus, the equation of motion for $s_{\mathbf{k}}$ is

$$\frac{d}{dt} s_{\mathbf{k}} = -\Gamma_{\mathbf{k}} \frac{\partial}{\partial s_{-\mathbf{k}}} H + \Gamma_{\mathbf{k}} \zeta(t).$$

where $\Gamma_{\mathbf{k}} = {\Gamma'_{\mathbf{k}}}^{-1}$. The variables $s_{\mathbf{k}}$ evolve as random functions of time. So their dynamics should be described using the theory of random processes. That is, these variables should be characterized in terms of distribution functions. The probability distribution $P(\{s_{\mathbf{k}}\}, t)$ satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \sum_{k \leq \Lambda} \Gamma_{\mathbf{k}} \frac{\partial}{\partial s_{\mathbf{k}}} \Big[\frac{\partial}{\partial s_{-\mathbf{k}}} H P \Big] + k_B T \sum_{k \leq \Lambda} \Gamma_{\mathbf{k}} \frac{\partial}{\partial s_{\mathbf{k}}} \frac{\partial}{\partial s_{-\mathbf{k}}} P.$$

In equilibrium, P is independent of time. It can be easily verified that the equilibrium solution is

$$P(\{s_{\mathbf{k}}\}) = \frac{1}{Z} \exp(-\frac{H}{k_B T}).$$

The stochastic model of dynamics outlined above provides the entire time development of the Fourier amplitudes leading to the final equilibrium state. However, the complete solution of the dynamical problem is much more involved than calculating the partition function in the equilibrium situation. In view of the fact that $< \zeta(t) >= 0$, the average value of $s_{\mathbf{k}}$ satisfies the equation

$$\frac{d}{dt} < s_{\mathbf{k}} >= -\Gamma_{\mathbf{k}} < \frac{\partial H}{\partial s_{-\mathbf{k}}} > .$$

Using the L-G hamiltonian (for $T > T_c$), one finds

$$\frac{d}{dt} < s_{\mathbf{k}} >= -2\Gamma_{\mathbf{k}}(a_2 + ck^2) < s_{\mathbf{k}} >,$$

This equation of motion shows that the relaxation time characterizing the time development of $s_{\mathbf{k}}$ is

$$\tau_k = \frac{1}{\Gamma_{\mathbf{k}}} \frac{1}{2(a_2 + ck^2)}.$$

The dependence of $\Gamma_{\mathbf{k}}$ on small values of $\mathbf{k}$ decides the relaxation time of the long wavelength modes. Assuming $\Gamma_{\mathbf{k}} = \Gamma$ to be a constant, one gets

$$\tau_k = \frac{1}{\Gamma} G(k),$$

where $G(k)$ is the correlation function for $T > T_c$,

$$G(k) = \frac{1}{2(a_2 + ck^2)} = \frac{\xi^2}{2c(1 + \xi^2 k^2)}.$$

The last step follows from the definition of the correlation length ξ for $T > T_c$. Employing the general scaling form for $G(k)$,

$$G(k) = \xi^{2-\eta} g(\xi k),$$

the relaxation time may be written as

$$\tau_k = \frac{1}{\Gamma} \xi^{2-\eta} g(\xi k).$$

Thus τ_0 diverges as

$$\tau_0 \sim |T - T_c|^{-\nu(2-\eta)},$$

if the system is in the vicinity of T_c during the time evolution. The divergence of τ_k is generally characterized in terms of an exponent z,

$$\tau_k = \frac{1}{\Gamma} \xi^z g(\xi k),$$

and in the quadratic approximation to the L-G model, $z = 2 - \eta = 2$ since $\eta = 0$.

8.1.1 RG for Dynamics

Near the critical point, the long wavelength modes relax very slowly. Just as there are spatial length scales of all sizes in a near critical system, time scales of all sizes are involved in the dynamics. So it is natural to seek an extension of the RG procedure to extract the basic features in dynamics. The RG steps for the static situation consist of eliminating the modes with k in the range Λ/q to Λ (k_B group), introducing a spatial rescaling by the scale factor q and renormalizing the remaining mode amplitudes with k in the range 0 to Λ/q (k_A group). In the dynamical situation, as the shorter wavelength modes are eliminated, their influences on time evolution also get removed. Note that the relaxation time of the shortest wavelength mode is the basic time unit in any dynamical description. So, the elimination of shorter wavelength modes should be accompanied by a rescaling of time at every step of the RG transformation. The rescaling parameter is related to the spatial scale factor q and it should reduce to unity when $q = 1$. The elimination of degrees of freedom in the dynamical situation would correspond to solving for $s_{\mathbf{k}}$ in the k_B group in terms of $s_{\mathbf{k}}$ in the k_A group and substituting them in the equations for $s_{\mathbf{k}}$ in the k_A group. This may also be done by integrating out the k_B group amplitudes from the Fokker-Planck equation for $P(\{s_{\mathbf{k}}\}, t)$ and thus obtaining a new Fokker-Planck equation with reduced number of independent variables. The other steps can be represented by the replacements

$$\begin{aligned} s_{\mathbf{k}}(t) &\rightarrow q^{1-\eta/2} s_{\mathbf{k}'}(t'), \\ \mathbf{k}' &= q\mathbf{k}, \\ t' &= tq^{-z}. \end{aligned}$$

The last step, which is the additional aspect of dynamic RG, is essentially the rescaling of the time unit. After these steps, it should be possible to write the remaining equations of motion as in the original model and thus extract the transformation equations for the parameter set $\mu = (a_2, c, \cdots, \Gamma)$ in the dynamical equations. The new parameter set μ' can be symbolically represented as $\mu' = \mathbf{R}_q \mu$, thus defining the RG transformation $\mathbf{R}_q$. The exponent z has to be chosen so as to obtain a non-trivial fixed point for the transformation equations.

The structure of the system during the time evolution can be probed experimentally by inelastic scattering techniques. The scattering cross-section Γ_{fi} (from a volume V) corresponding to a wave vector change k and energy change $\hbar\omega$ can be expressed as

$$\Gamma_{fi} \sim V \int_0^\infty C(k,t) \exp(\imath\omega t) dt,$$

where $C(k,t)$ is the time dependent correlation function defined as

$$C(k,t) =< s_{\mathbf{k}}(t)s_{-\mathbf{k}}(0) > .$$

The equivalence of the two models characterized by the parameter sets μ and μ' yields the functional equation

$$C(k,t,\mu) = q^{2-\eta}C(qk, q^{-z}t, \mu').$$

This is a generalization of the equation for the static correlation function $G(k,\mu)$. Analysis similar to that for $G(k,\mu)$ leads to a scaling form

$$C(k,t,T) = \xi^{2-\eta}g_c(\xi k, \xi^{-z}t),$$

for T close to T_c. The function g_c depends only on the scaled variables ξk and $\xi^{-z}t$. This form suggests that for a specific value of k, the time scale of evolution characterized by the relaxation time τ_k must be of the general form

$$\tau_k = \xi^z g(\xi k),$$

which is same as that obtained earlier. Derivation of the RG transformation equations for the quadratic L-G model can be done easily and one gets $z = 2$. The same result is obtained even with the general L-G model for $d \geq 4$. The ϵ expansion for $d < 4$ yields

$$\begin{aligned} z &= 2 + a\eta + O(\epsilon^3), \\ a &= 6\ln(\frac{4}{3}) - 1. \end{aligned}$$

Thus one finds that the exponent z is not the same as $2+\eta$ as suggested from the relaxation time for $< s_{\mathbf{k}} >$.

8.2 Percolation Phenomena

Percolation of any physical quantity (or property) involves the establishment of a geometrical connectedness in a system with random spatial structure. Fluid flow through a network of pores in a medium, where a certain fraction of the pores are blocked in a random manner, provides an example of percolation. For simplicity, one may assume that the bonds in a large 3-D lattice represent the pores and that each pore (or bond) is open with certain pre-assigned probability p. Thus, on an average, a fraction p of the pores is open and the remaining fraction $(1-p)$ is blocked. If the fluid is allowed to enter the medium on one of its faces at a constant rate, then in cases when $p \ll 1$, it can not easily pass through the medium to the opposite face. Thus for small values of p, fluid can not percolate through the medium. Of course for $p \approx 1$,

Table 8.1: Percolation Thresolds

Lattice	p_c $(site)$	p_c $(bond)$
$2-D\ Square$	0.593	0.500
$2-D\ Triangular$	0.500	0.347
$3-D\ Cubic$	0.312	0.249
$3-D\ BCC$	0.245	0.179
$3-D\ FCC$	0.198	0.119

fluid will easily flow through the medium. The experimental observation is that there is a critical value of $p = p_c$ such that for $p \leq p_c$, the medium obstructs the fluid flow while for $p > p_c$ there is a finite flow rate across the medium. For unobstructed fluid flow, there should exist a connected path of open pores across the medium and so it may be concluded that such a path appears in an abrupt manner at p_c. The random network just described has also been used to model the gelation transition observed in the solutions of certain organic materials. Here one can imagine the lattice sites as the molecules of the compound and the bonds between neighbouring sites as the random links between them. If p is probability of formation of a link, then at p_c, a network of connected molecules indicating the onset of gelation, will exit in the system.

There is also a formulation of percolation phenomena in terms of site percolation models. Imagine that the sites of a lattice are occupied by particles randomly and independently. If p is the probability that a particular site is occupied, then for small values of p there are mostly isolated particles and rarely small clusters of particles in the system. As the value of p is increased, larger and larger clusters of particles would appear. At a critical value of $p = p_c$, a cluster spanning the entire domain appears in the system. The random structure of occupied sites is a good model of a disordered structure and has been extensively used to study diffusion on disordered lattices. The percolation threshold (p_c) values depend on the type and dimension (d) of the lattice and a small list corresponding to the infinite lattice limit is given in Table 8.1. All the approximate results are obtained by simulating the percolation model on a computer. The information about the random occupation of sites (or bonds) of a large lattice at a specific value of p are stored in the computer memory and then the cluster sizes are analyzed at different values of p. Thus one can determine the value of p_c at which a cluster spanning the entire lattice is obtained for the first time. The results so obtained for various lattice sizes are then extrapolated to the infinite lattice limit.

Several physical quantities pertaining to the cluster structure show very distinct behaviour when p is close to p_c. The probability P that an arbitrary site (or bond) in the lattice belongs to the infinite cluster is one of these quantities. It is clear that $P = 0$ for $p \leq p_c$ while it is found that P varies as a power law for $p > p_c$,

$$P(p) \approx (p - p_c)^{\beta},\ p \geq p_c.$$

The exponent β characterizes the way P vanishes as p approaches p_c from above and is analogous to the order parameter exponent for magnetic transition. The physical properties of a random structure which depend on the presence of the geometric connectedness are related to P. Thus the elastic properties of gels or the conductivity in a random network are related to the probability P. Another useful quantity to characterize percolation behaviour is the mean cluster size S. For $p < p_c$, clusters are of finite sizes, however, as an infinite cluster appears at p_c, one expects S to diverge as $p \rightarrow p_c$ from below. The divergence is quantified in terms of an exponent γ as

$$S(p) \approx (p - p_c)^{-\gamma},$$

and γ is similar to the magnetic susceptibility exponent. The divergence of S is indicative of the unbounded increase of some typical average linear size of clusters. This length scale, ξ, usually called the correlation length, behaves as

$$\xi(p) \approx |p - p_c|^{-\nu},$$

as observed in thermal critical phenomena. The total number of clusters M in the lattice depends on p and the non-analytic part of M is found to vary as

$$M'(p) \approx (p - p_c)^{2-\alpha}.$$

The quantity M' is similar to the singular part of free energy density in thermal systems and its second derivative diverges like specific heat in thermal critical behaviour. All the exponents introduced are not independent and scaling hypotheses have been advanced to derive relations among them. Two scaling laws are

$$\begin{aligned} \alpha + \beta + \gamma &= 2 \\ 2 - d\nu &= \alpha \end{aligned}$$

where d is the spatial dimension. Some typical values of the exponents are shown in Table 8.2. The two exponents σ and τ given in the table are introduced to quantify the variation of the number of clusters n_s (per lattice site) of size s. For large values of s, the scaling form proposed for n_s is

$$n_s \approx s^{-\tau}\ f[(p - p_c)s^{\sigma}],$$

Table 8.2: Percolation Model Exponents

Exponent	$d = 2$	$d = 3$	*Bethe Lattice*
β	0.14	0.4	1.0
γ	2.39	1.8	1.0
α	-0.66	-0.6	-1.0
ν	1.33	0.9	0.5
σ	0.4	0.45	0.5
τ	2.05	2.2	2.5

where f is a function of the scaled variable $(p - p_c)s^\sigma$. The exponents τ and σ are related to β and γ as

$$\sigma = \frac{1}{\beta + \gamma}$$
$$\tau = 2 + \frac{\beta}{\beta + \gamma}$$

The entries in the last column of the table are values of the exponents for the Bethe lattice which is constructed as follows. Starting from an arbitrary point, one draws z bonds. Then at the vertices of each bond, $z - 1$ bonds are drawn and this process is repeated again and again. However, the bonds never cross each other. Thus the Bethe lattice has a tree like structure, each branch breaks into $z - 1$ branches. An important simplicity of the Bethe lattice is that there is one and only one path between any two vertices in the lattice. This feature makes it possible to solve the percolation problem exactly and p_c is $(z - 1)^{-1}$. The Bethe lattice is an approximate model of geometrical connectivity in the percolation problems. It neglects the details of fluctuations in connectivity and is similar to the mean field models in thermal critical phenomena. An estimate of the magnitude of fluctuations show that this model is exact for $d \geq 6$.

The similarity between thermal critical phenomena and percolation behaviour is very striking. In the former, at a given temperature, a configuration of 'particles' occurs in the system with probability given by the Boltzmann factor $\exp(-E/k_BT)$ where E is the energy of the configuration. In percolation models, the site or a bond of the lattice is occupied in a random way with probability p. Thus temperature T and p are analogous parameters. The divergence of the correlation length (or mean linear size of clusters) is common in both phenomena and for $p \approx p_c$, clusters of all sizes exist in the system. The renormalization group idea of progressively incorporating length scales of all sizes is hence expected to be useful in percolation phenomena also.

8.2.1 Simple RG Calculations

Some illustrative applications of the real space renormalization group calculations are discussed below. Just like in thermal critical phenomena, sites (or bonds) are grouped together to form a cell lattice. Then with a proper definition for cell occupation, a relation connecting the occupation probabilities p' and p at the cell and site levels is derived. The transformation equation yields the percolation threshold and the exponent ν. If the linear size of the cell is q (in units of the site spacing), the correlation lengths ξ and ξ' (measured in appropriate lattice spacing units) of the two models are related as

$$\xi'(p') = \frac{1}{q}\xi(p).$$

If the transformation law relating p and p' is

$$p' = R(p),$$

then its non-trivial fixed point is identified with the percolation threshold p_c, that is

$$p_c = R(p_c).$$

Near the fixed point, the transformation can be linearized as

$$\Delta p' = (\frac{\partial R}{\partial p})^* \Delta p = q^y \Delta p,$$

for the deviations of p and p' from the fixed point. The same analysis for the magnetic case yields the functional equation for the correlation length,

$$\xi(\Delta p) = q\xi(q^y \Delta p).$$

Its solution for arbitrary q is

$$\xi(\Delta p) = |\Delta p|^{1/y} \xi(1),$$

so that the exponent $\nu = 1/y$. Consider the site percolation problem on a 2-D triangular lattice shown in Figure 8.1. The sites at the vertices of a triangle are grouped together to form a cell. The cell lattice is again triangular but the spacing is $\sqrt{3}$ times larger than the site spacing. So $q = \sqrt{3}$. Now, a cell is taken to be occupied if there exits a connection between its opposite sides and otherwise it is defined to be empty. This rule takes the place of the majority rule in the case of Ising model. For percolation, connectivity of the network is the important aspect and hence the majority rule is modified in this manner. Now, note that p^3 is the probability that all sites in a cell are occupied and then a connection across the sides of the cell definitely exists. However, a

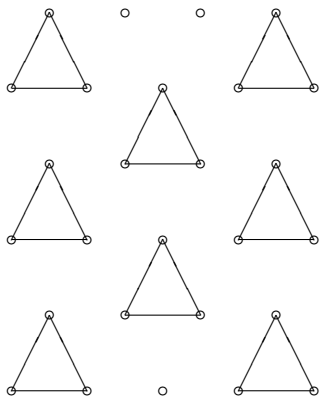

Figure 8.1: Cells in a Triangular Lattice.

connection can also exist if any two sites are occupied and the probability for the same is $3p^2(1-p)$, the factor 3 being the number of ways of choosing two occupied sites out of three. If just one site is occupied, there is no connection between the sides of the cell. Therefore the probability p' of cell occupation is

$$p' = p^3 + 3p^2(1-p).$$

The fixed point values of the transformation are $p^* = 0,\ 1/2$ and 1. The first and last values are trivial fixed points. The non-trivial fixed point value $1/2$ coincides with the percolation threshold given in Table 8.1. On linearising the transformation one gets

$$\Delta p' = \frac{3}{2}\Delta p,$$

which yields the exponent $\nu \approx 1.355$ in good agreement with the known result. Such accurate results are not always obtained for different types of lattices and dimensions. In a 2-D square lattice, let a cell be formed (see Figure 8.2) by grouping together 4 sites so that the scale factor is $q = 2$. Further, define that a cell is occupied if its left and right sides are connected together. The probability p' is then given by

$$p' = p^4 + 4p^3(1-p) + 2p^2(1-p)^2.$$

The first term arises if all the four sites are occupied. The second term corresponds to the four ways of choosing three occupied sites and one empty site. Similarly, the last term is obtained when two sites are occupied and two are

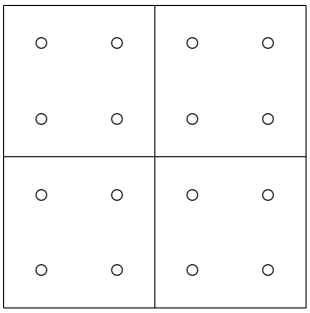

Figure 8.2: Cells in a Square Lattice.

empty. The non-trivial fixed point value yields $p_c \approx 0.62$ and $\nu \approx 1.602$. Thus the agreement is much less than in the case of triangular lattice. The general conclusion derived from such calculations is that it is necessary to consider large cells for obtaining accurate results and so the technique has to be coupled with computer simulation methods.

8.3 Polymer Conformations

Polymers are long flexible organic molecules in which a specific chemical group, usually called a monomer, is repeated a large number of times. For example, in polystyrene, the chemical group consists of the $CH_2 - CH$ bond and a Benzene molecule and is repeated as much as 10^5 times. The extraction of the global features of such objects has been one of the impressive successes of modern statistical physics.

One of the simplest models employed to describe the various conformations of a polymer chain in a solvent is the self-avoiding random walk (SAW) model. It is a modified form of the simple random walk model of Brownian motion. Imagine that the random walk starts at some point in a d-dimensional lattice. The walker can jump to the nearest neighbour sites with equal probability and a particular realization of the walk with N steps is taken as a possible polymer conformation with N units. In the SAW, there is a restriction that the walker can visit a lattice site only once. This restriction is imposed to model the

physical fact that a polymer chain can not cross over itself. Imposing the self-avoiding restriction leads to serious mathematical difficulties in describing the SAW and many important results on its global properties have been obtained via computer simulation techniques.

A quantity of primary interest is the number of walks of N steps starting from an arbitrary origin o and ending at the site i. If this number is denoted as $\Gamma_N(oi)$, the probability that the walker is at site i after N steps (from o) can be expressed as

$$P_N(oi) = \frac{\Gamma_N(oi)}{\Gamma_N^t},$$

where

$$\Gamma_N^t = \sum_i \Gamma_N(oi),$$

is the total number of possible N step walks starting from the origin o. In the simple random walk model, the walker can visit any of the z nearest neighbour sites at each step, and hence

$$\Gamma_N^t = z^N,$$

where the coordination number z has values 4 and 6 for a simple cubic lattice in 2 and 3 dimension respectively. Due to the self-avoiding character, this result is modified for SAW and in the limit of large N one finds that

$$\Gamma_N^t \sim z_0^N N^{\gamma-1},$$

where the parameter z_0 (called the effective coordination number, $z_0 \approx 4.7$ for 3-D simple cubic lattice) is less than z and γ is a universal exponent depending only on the spatial dimension. Another important parameter is the number of N step walks ending at one of the nearest neighbours of the starting point o. If it is denoted as $\Gamma_N(n.n)$, its asymptotic variation is found to be

$$\Gamma_N(n.n) \sim z_0^N N^{-2+\alpha},$$

where α is another universal number. The end to end length of a walk can be written as

$$\mathbf{R}_N(oi) = \mathbf{r}_{o1} + \mathbf{r}_{12} + \mathbf{r}_{23} + \cdots + \mathbf{r}_{N-1\ i},$$

where $\{\mathbf{r}_{jk}\}$ are vectors along the lattice bonds between the sites j and k. The root mean square value of $\mathbf{R_N}(oi)$ for large N varies as

$$R \sim a_0 N^{\nu},$$

where a_0 is the lattice spacing and ν is yet another exponent. For the simple random walk, the exponent ν is $1/2$ for all values of d. R is similar to the correlation length ξ in thermal critical phenomena and its power law divergence

Table 8.3: Exponents for SAW

Exponent	$d=1$	$d=2$	$d=3$	*Random Walk*
γ	1.0	1.33	1.17	1.0
ν	1.0	0.74	0.6	0.5
α	1.0	$\cdots$	$\cdots$	0.0

for large N is analogous to the divergence of ξ near the critical temperature T_c. Some typical values of the exponents are given in Table 8.3. Note that the mean end to end distance increases faster than that in the case of simple random walk and this fact is a manifestation of the restriction that any site can not be visited more than once. Thus there is a tendency for the walk to move away faster than in the case of the simple walk.

A rather strong connection between the SAW model of polymer conformations and thermal critical phenomena was brought out by P. G. de Gennes. He showed that the spin-spin correlation function for the n-vector model in the limit $n \to 0$ can be written as

$$\lim_{n\to 0} < \mathbf{s}_o \mathbf{s}_i >= \sum_N \Gamma_N(oi) \Big[\frac{J}{k_B T}\Big]^N, \tag{8.1}$$

where $\Gamma_N(oi)$ is the number of N step SAWs from o to i and J/k_BT is the reduced coupling strength of the n-vector model. Even though the $n \to 0$ limit is a pure mathematical device, the above result relates, in some sense, the magnetic critical behaviour with SAWs. Eq.(8.1) can be rewritten as

$$\lim_{n\to 0} < \mathbf{s}_o \mathbf{s}_i > \sim \sum_N P_N(oi) N^{\gamma-1} \Big[\frac{J z_0}{k_B T}\Big]^N,$$

by using the definition of the probability $P_N(oi)$ and the asymptotic form of Γ_N^t discussed earlier. If the critical temperature T_c is defined as

$$T_c = \frac{J z_0}{k_B},$$

then for a small deviation ΔT from T_c one gets

$$\frac{J z_0}{k_B T} \approx 1 - \frac{\Delta T}{T_c} \quad \approx \exp\Big[-\frac{\Delta T}{T_c}\Big].$$

Then the correlation function becomes

$$\lim_{n\to 0} < \mathbf{s}_o \mathbf{s}_i > \sim \sum_N P_N(oi) N^{\gamma-1} \exp\Big[-\frac{\Delta T}{T_c} N\Big]. \tag{8.2}$$

Eq.(8.2) shows that N and ΔT are some sort of conjugate variables. The summation over N may be replaced by an integration and then the probability $P_N(oi)$ can be obtained by a Laplace transform inversion of the correlation function. With this interpretation, the exponents of SAW can be obtained from those of the n-vector model in the limit $n \to 0$. This connection also shows that, for $d \geq 4$, the Landau's theory exponents of critical phenomena (which are independent of n and d) are identical to those of SAW. In fact the exponents for the simple random walk are same as those of Landau's theory. The interesting conclusion so obtained is that for $d \geq 4$, the universal aspects are not modified by the restriction imposed in SAW.

8.3.1 Decimation of the Chain

The following analysis, due to de Gennes, is an application of the RG approach to the problem of chain conformations. A polymer chain is characterized by the length of the monomer unit and a repulsive interaction between monomers which prevents them from crossing over. In the SAW model, the lattice spacing a_0 and the self-avoiding restriction represent these features. One may associate a monomer unit with each site visited in the SAW. Then the self avoiding restriction may be modeled in terms of an excluded volume parameter v_0 between two monomers. In the case of two hard sphere particles, v_0 is eight times the particle volume. Now, imagine that every q monomers along the chain are grouped together to form a coarse description of polymer conformations and let (a_1, v_1) denote the parameter set in that description. The length unit a_1 is the mean end to end distance of q monomers and v_1 is the excluded volume parameter of two coarse units. In the absence of self-avoiding restriction, it is clear that $a_1 = a_0 q^{1/2}$. So an equation relating a_1 and a_0 may be written as

$$a_1 = a_0 q^{1/2} f_1(q, a_0, v_0),$$

where the correction factor $f_1 \geq 1$, arising from the self-avoiding restriction, must approach unity as $v_0 \to 0$. The dependence of f_1 on a_0 and v_0 must be in terms of the dimensionless parameter $u_0 = v_0/a_0^d$ (d=spatial dimension) since a_0 is the basic length unit. Then a_1 can be expressed as

$$a_1 = a_0 q^{1/2} f(q, u_0).$$

There are a total of q^2 monomer pairs between two coarse units and hence v_1 is $q^2 v_0$ in the case of free monomers. Due to the extended growth of the chain, all the pairs do not come close enough to experience the excluded volume restriction and hence v_1, which is less than this limiting value, can be expressed as

$$v_1 = q^2 v_0 g(q, u_0).$$

The correction factor $g \leq 1$ is also expressed in terms of u_0. This relation can be rewritten in terms of the dimensionless parameter $u_1 = v_1/a_1^d$ as

$$u_1 = q^{2-d/2} u_0 h(q, u_0),$$

where the factor $h = g/f^d$ is less than unity. If the two descriptions are equivalent, the mean end to end distances $\bar{R}_0$ and $\bar{R}_1$ expressed in terms of a_0 and a_1 respectively are related as

$$\bar{R}_0(N, u_0) = \frac{a_1}{a_0} \bar{R}_1(\frac{N}{q}, u_1).$$

The factor a_1/a_0 is just the spatial rescaling parameter and the number of coarse units is N/q. On repeating the process of coarse graining l times one gets

$$\bar{R}_0(N, u_0) = \frac{a_l}{a_0} \bar{R}_l(\frac{N}{q^l}, u_l).$$

As the value of l increases, the number of units in the renormalized chain reduces. This fact implies that the parameter u_l defined by

$$u_l = q^{2-d/2} u_{l-1} h(q, u_{l-1}),$$

will approach a constant value u^*. Note that it is the parameter u and not v which approaches a limit for large l. Then the transformation equation for a_l reduces to

$$a_l = a_{l-1} q^{1/2} f(q, u^*),$$

and can be expressed as

$$\begin{aligned} a_l &= a_{l-1} q^{\nu}, \\ \nu &= \frac{1}{2} + \frac{\ln(f)}{\ln(q)}. \end{aligned}$$

So $\bar{R}_0$ takes the form

$$\bar{R}_0(N, u_0) \sim q^{l\nu} \bar{R}_l(\frac{N}{q^l}, u^*).$$

Now, choosing l such that $q^l = N$ one finds that

$$R_0(N, u_0) = a_0 \bar{R}_0 \sim \ \textit{constant}\ N^{\nu},$$

which is the observed power law divergence. For $d \geq 4$, the recursion relation shows that $u_l \to u^* = 0$ since both factors $q^{2-d/2}$ and h are less than unity. Hence $f(q, u^*) = 1$ and $\nu = 1/2$ for $d \geq 4$. Thus the results of simple random walk model (valid for $d \geq 4$) and Landau's theory of thermal phase transitions are similar.

8.3.2 RG Calculation of ν

The similarities in thermal critical phenomena and polymer conformations suggest that the universal aspects of the latter in the large N limit is a consequence of the existence of the fundamental length scale R. In fact, in the SAW model, walks of all sorts of end to end distance exist and thus the problem has again many length scales. This aspect together with the formal connection to thermal critical behaviour have prompted the application of RG ideas to the problem of SAW conformations. Summing over the final end point i in Eq.(8.2) and using the normalization of the probability $P_N(oi)$, one gets

$$\chi(T) = \frac{1}{T}\sum_i \lim_{n\to 0} < \mathbf{s}_o\mathbf{s}_i > \sim \frac{1}{T}\sum_N N^{\gamma-1}\exp(-\frac{\Delta T}{T_c}N) \sim (\frac{\Delta T}{T_c})^{-\gamma},$$

where the definition of susceptibility $\chi(T)$ in terms of the spatial sum of the correlation function has been used. The last step in this equation is obtained by replacing the sum over N by an integral and calculating its dominant contribution. This result shows that the exponent γ in the definition of Γ_N^t is analogous to the susceptibility exponent for magnetic case. Therefore, summing over the possible values of i, Eq.(8.1) can be written as a generating function for the SAW conformations. That is

$$G(\kappa) = \sum_N \kappa^N \Gamma_N^t,$$

where the parameter $\kappa = J/k_BT$. The divergence of $\chi(T) = G(\kappa)/T$ at T_c can then be expressed as

$$G(\kappa) \sim |\kappa - \kappa_c|^{-\gamma}, \tag{8.3}$$

where the critical value $\kappa_c = J/k_BT_c = z_0^{-1}$, z_0 being the effective coordination number for SAW. From the scaling form of the spatial integral of correlation function, it is known that

$$\chi(T) \sim \xi^{2-\eta},$$

where ξ is the correlation length. Using this result, $G(\kappa)$ can be expressed in terms of the mean end to end distance R as

$$G(\kappa) \sim R^{2-\eta}.$$

Combining with Eq.(8.3) and using the scaling relation $\gamma = (2-\eta)\nu$, one finds

$$R(\kappa) \sim |\kappa - \kappa_c|^{-\nu}.$$

Thus the divergence of R for SAW can also be expressed in terms of the parameter κ. This parameter, introduced in Eq.(8.3), is similar to the fugacity

Figure 8.3: Coarse Graining SAW Steps.

appearing in the definition of the grand partition function. κ^N is a weight factor for the number of N step walks. One may therefore assign a weight κ to every step of the walk.

A simple illustration of the use of RG ideas to calculate ν is the following. A coarse description of SAW is obtained by grouping a certain number of steps in the walk. In this way, a relation between the fugacity parameters κ' and κ in the two descriptions is obtained. Then the non-trivial fixed point $\kappa_c = z_0^{-1}$ and the exponent ν are found in the usual manner. Consider the eight bonds of the lattice cell shown in Figure 8.3 where each bond has a weight factor κ. The renormalized bonds with weight factors κ' are shown on the right. If there exists a self-avoiding path across the cell (from left to right), the renormalized bond also is taken to be a SAW step. A similar criterion is adopted for the vertical direction as well. Further, it is assumed that the SAW steps start from the left-bottom corner. Due to the symmetry of the problem, it is then enough to consider the six steps marked in Figure 8.3 to obtain the strength of κ'. Thus the four step walk $(2, 3, 4, 6)$ yields a contribution κ^4 in κ'. Then, two types of three step walks, $(2, 3, 5)$ and $(1, 4, 5)$, contribute a term $2\kappa^3$. Finally, there is a two step walk $(1, 6)$ and the transformation is

$$\kappa' = \kappa^4 + 2\kappa^2 + \kappa.$$

The three fixed points of the transformation are 0, $0.466\cdots$ and ∞. Taking $\kappa_c \approx 0.446$, one gets $z_0 \approx 2.146$ while the simulation result for 2-D square lattice is ≈ 2.639. Linearizing the transformation at κ_c, one finds that the slope is $\approx 2.639 \approx 2^{1.4}$ and hence $\nu \approx 0.715$ which may be compared with the estimated value of 0.75. Similar calculations for 3-D simple cubic lattice yield $z_0 \approx 4.68$ and $\nu \approx 0.588$ which are again in good agreement with simulation results.

8.4 Chaotic Maps

Bifurcation and transition to chaos in nonlinear dynamics is another area of application of RG ideas. Even simple one dimensional maps, which model

nonlinear evolution, exhibit certain interesting scaling and universal properties. A 1-D map is a transformation of a point x_n on the real axis to another point x_{n+1} and is defined by

$$x_{n+1} = f(x_n, \lambda), \quad n = 0, 1, \cdots, \tag{8.4}$$

where f is a nonlinear function and λ is a parameter. Such transformations occur in numerical algorithms to determine zeros of a given function. They are also used as simple models of population growth and stabilization. In the latter case, if a steady state population exists, it is given by the fixed point of the transformation

$$x^* = f(x^*, \lambda).$$

The nature of the fixed point depends on the function f. If the magnitude of the slope, $|f'(x^*)|$ is less than unity, then starting with any arbitrary point x_0 in the neighbourhood of x^*, the iterates for large n tend to x^* and it is said to be a stable fixed point. x^* is marginally stable if $|f'(x^*)| = 1$ and unstable if $|f'(x^*)| > 1$. For definiteness, consider the map (called logistic map in population dynamics models) defined as

$$x_{n+1} = \lambda x_n(1 - x_n), n = 0, 1, \cdots.$$

From a graph of the corresponding f, it is easily concluded that if any iterate lies outside the interval [0,1], then the later iterates asymptotically tend to $-\infty$. The maximum of f is $\lambda/4$ at $\bar{x} = 1/2$. Therefore, for non-trivial dynamics it is enough to consider x in [0,1] and λ in [0,4]. Now, $x_0^* = 0$ is a trivial fixed point and it is stable for $0 \leq \lambda < 1$ since $f'(0) = \lambda$. Thus if λ is increased continuously from 0, at the value 1, $x^* = 0$ looses stability. However for $1 < \lambda < 3$, there is a non-trivial fixed point $x_1^* = 1 - 1/\lambda$ since $f'(x_1^*) = 2 - \lambda$. For $\lambda = 2$, the fixed point $x_1^* = 1/2$ is said to be super stable since $f'(1/2) = 0$. When λ is increased from 1 to 3, $f'(x_1^*)$ decreases from 1 to -1. For λ slightly greater than 3, x_1^* becomes unstable. But a 2-point limit cycle defined by

$$\begin{aligned} x_{22}^* &= f(x_{21}^*, \lambda), \\ x_{21}^* &= f(x_{22}^*, \lambda), \end{aligned}$$

becomes stable. In fact x_{2k}^* $(k = 1, 2)$ are the fixed points of the second functional iterate $f_2(x)$ since

$$\begin{aligned} x_{21}^* &= f(f(x_{21}^*)) \equiv f_2(x_{21}^*), \\ x_{22}^* &= f(f(x_{22}^*)) \equiv f_2(x_{22}^*). \end{aligned}$$

The n^{th} functional iterate $f_n(x)$ is defined as

$$f_n(x) = f(f(f \cdots (x)).$$

For the logistic map, the second iterate is

$$f_2(x) = \lambda^2 x(1-x)(1-\lambda x(1-x)).$$

It has a minimum at $\bar{x} = 1/2$ and a maximum on its either sides. The fixed point equation for $f_2(x)$ can be factored as

$$(\lambda - 2 - \lambda y^*)(\lambda^2 y^{*2} - 2\lambda y^* - \lambda^2 + 2\lambda + 4) = 0,$$

where $y^* = 2x^* - 1$. The root $y^* = 1 - 2/\lambda$ yields $x^* = 1 - 1/\lambda$ and is unstable. The relevant fixed points are

$$x^*_{2k} = \frac{1}{2} - \frac{1}{2\lambda}\{1 - (-1)^k(\lambda^2 - 2\lambda - 3)^{1/2}\}, \quad k = 1, 2.$$

The derivatives of f_2 at these fixed points are

$$f_2'(x^*_{21}) = f_2'(x^*_{22}) = f'(x^*_{21})f'(x^*_{22}) = 4 + 2\lambda - \lambda^2.$$

Thus f_2' decreases from 1 to -1 as λ is varied from 3 to $1 + \sqrt{6}$. It is zero for $\lambda = 1 + \sqrt{5}$ and the 2-point limit cycle is super stable. At this point, x^*_{21} is at the abscissa of the minimum of f_2 and x^*_{22} is at the abscissa of the maximum on the right. The break up of the fixed point x^*_1 to a 2-point limit cycle is called a pitchfork bifurcation. When λ is slightly greater than $1 + \sqrt{6}$, the 2-point limit cycle becomes unstable but a 4-point limit cycle determined by the fixed points of $f_4(x)$ becomes stable. As λ is increased further, this general pattern of a 2^n-point cycle becoming unstable at Λ_n and a 2^{n+1}-point cycle acquiring stability continues. Further there exists a parameter value λ_n at which the 2^n-point cycle is super stable with the fixed points at the extreme of $f_{2^n}(x)$. The smallest member has the value 1/2 which is the maximum of $f(x)$ itself. Note that according to the definition, $\Lambda_0 = 3$, $\Lambda_1 = 1 + \sqrt{6}$, $\lambda_0 = 2$ and $\lambda_1 = 1 + \sqrt{5}$. The parameter values Λ_n accumulate to λ^* and for λ greater than λ^*, limit cycles with odd number of periods and aperiodic iterates occur. Beyond λ^*, the dynamics generated by the map is said to be chaotic. For the logistic map $\lambda^* \approx 3.569943\cdots$. A schematic representation of the period doubling bifurcation is shown in Figure 8.4.

8.4.1 RG Theory

There are two universal numbers discovered by Feigenbaum in connection with period doubling bifurcation of a family of 1-D maps. The first is the rate of bifurcation δ defined as

$$\Lambda_n \sim \lambda_\infty - \frac{A}{\delta^n},$$

where A is a constant and $n \gg 1$. The parameter λ_n, at which the 2^n-point cycle is super stable, also converges to λ_∞ in the same manner since

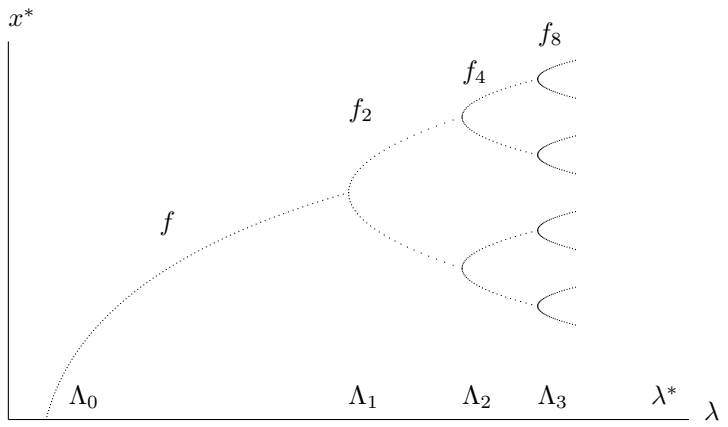

Figure 8.4: Bifurcation Diagram.

it is bracketed by Λ_{n-1} and Λ_n. The second universal number arises from the observation that the functional iterates $f_{2^{n-1}}(x, \lambda_n)$ are self similar. Note that at these λ values, one of the fixed points of f_{2^n}, say $x^*_{2^n 1}$, coincides with the abscissa $\bar{x} = 1/2$ corresponding to the maximum of $f(x)$. The function $f(x, \lambda_1)$ for $\bar{x} = x^*_{21} \leq x \leq x^*_{22}$ and $f_2(x, \lambda_2)$ for $x^*_{42} \leq x \leq x^*_{41} = \bar{x}$ can be superposed by properly scaling x and the amplitude. The pairs $(\bar{x}, x^*_{22})$ and $(\bar{x}, x^*_{42})$ form 2-point cycles of $f(x, \lambda_1)$ and $f_2(x, \lambda_2)$ respectively. So these functions in the regions $\bar{x} \leq x \leq x^*_{22}$ and $x^*_{42} \leq x \leq \bar{x}$ respectively, are exactly contained by square boxes, of sides $|d_1| = |x^*_{22} - \bar{x}|$ and $|d_2| = |x^*_{42} - \bar{x}|$ shown in Figure 8.5. Further, from the chain rule of differentiation, it follows that if $f(x)$ has a power law behaviour near $\bar{x}$, then $f_2(x)$ also has the same behaviour at $\bar{x}$. Hence the scaled functions $d_1^{-1} f(x d_1, \lambda_1)$ and $d_2^{-1} f_2(x d_2, \lambda_2)$ can be superposed on one another. The rescaling parameters d_n, which make the functions $d_n^{-1} f_{2^{n-1}}(x d_n, \lambda_n)$ self similar, approach zero for large n and

$$\lim_{n \to \infty} |\frac{d_n}{d_{n+1}}| = \alpha,$$

where α is a universal number. The universality of δ and α is that they depend only on the order of maximum z of $f(x)$ near $\bar{x}$. That is, for all unimodal functions, $f(x)$ mapping the interval [0,1] on to itself and varying as

$$f(x) \sim a_0 + a_z (x - \bar{x})^z, \quad z = 2, 4, \cdots,$$

near $\bar{x}$, δ and α depend only on z. For the logistic map ($z = 2$) Feigenbaum obtained the values $\delta \approx 4.669201 \cdots$ and $\alpha \approx 2.502907 \cdots$. It is nice to note that three steps are involved in obtaining the successive self similar functions. They are, shifting of λ_n to λ_{n+1}, changing the scale factor d_n to d_{n+1} and

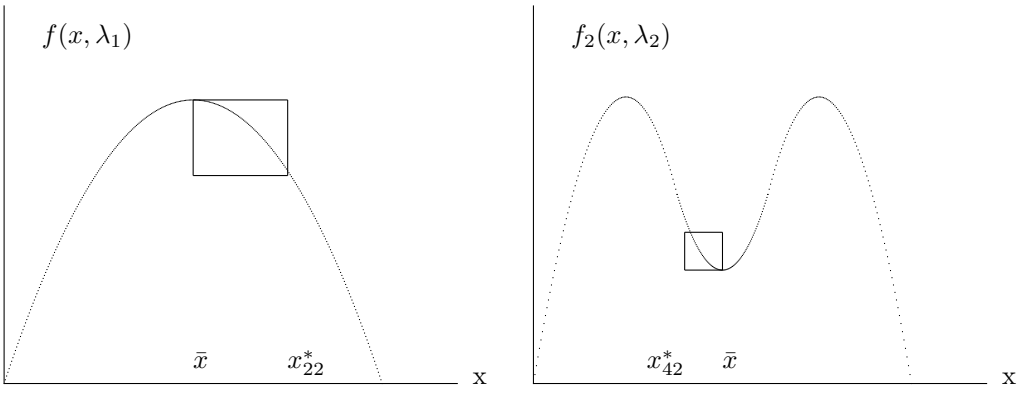

Figure 8.5: Self Similar Functions.

changing the amplitude d_n^{-1} to d_{n+1}^{-1}. These steps are analogous to those in the RG theory of critical behaviour.

Feigenbaum also showed that the limiting function

$$g^* = \lim_{n\to\infty}(-\alpha)^n f_{2^n}\Big(x/(-\alpha)^n, \lambda_{n+1}\Big),$$

is also universal and depends only on z. In fact there is a hierarchy of universal functions defined as

$$g_r(x) = \lim_{n\to\infty}(-\alpha)^n f_{2^n}(x/(-\alpha)^n, \lambda_{n+r}), \quad r = 0, 1, \cdots,$$

and $g^*(x)$ is simply $g_1(x)$. The definition implies that

$$g_{r-1}(x) = -\alpha g_r(g_r(x/\alpha)). \tag{8.5}$$

This functional recursion, which involves the steps of rescaling x and changing the amplitude, is analogous to Wilson's recursion formula for $Q(s)$ discussed in the Chapter 5. The limiting form of g_r,

$$g(x) = \lim_{r\to\infty} g_r(x),$$

is a fixed point of the recursion in Eq.(8.5) and satisfies

$$g(x) = -\alpha g(g(x/\alpha)). \tag{8.6}$$

Its solution $g(x)$, with the boundary condition $g(0) = 1$, also yields $\alpha = g(1)^{-1}$. For large values of r, the deviation of $g_r(x)$ from $g(x)$,

$$y_r(x) = g(x) - g_r(x),$$

can be obtained by linearising Eq.(8.5) around $g(x)$. The result is the linear functional equation

$$y_{r-1}(x) = -\alpha y_r(g(x/\alpha)) - \alpha g'(g(x/\alpha))y_r(x/\alpha).$$

The substitution

$$y_r(x) = \mu^{-r}\phi(x),$$

yields an eigenvalue equation

$$\Omega\phi(x) = -\alpha\phi(g(x/\alpha)) - \alpha g'(g(x/\alpha))\phi(x/\alpha) = \mu\phi(x),$$

In addition to the spectrum $|\mu| < 1$, the operator Ω is found to have a unique positive eigenvalue $\mu = \delta$ where δ is the bifurcation rate. Thus Eq.(8.6) describes all the universal aspects of period doubling bifurcation of 1-D maps.

The analogy between period doubling bifurcation and critical behaviour can be extended further. If $N(\Lambda_n)$ denotes the length of the limit cycle at λ slightly less than Λ_n, then

$$N(\Lambda_n) = 2N(\Lambda_{n-1}). \tag{8.7}$$

This relation is similar to that connecting the correlation lengths of two equivalent models of critical behaviour. The factor 2 is like the spatial rescaling factor and Λ_n and Λ_{n-1} are similar to the parameters characterizing the two models. As $\Lambda_n \to \lambda^*$, $N(\Lambda_n)$ diverges as 2^n and this behaviour is analogous to the divergence of the correlation length as the critical point is approached. Eq.(8.7) implies the existence of a recursion relation of the type

$$\Lambda_n = R(\Lambda_{n-1}),$$

between the bifurcation points Λ_n and Λ_{n-1}. Then the accumulation point λ^* is the fixed point defined by

$$\lambda^* = R(\lambda^*).$$

For small deviations from λ^*, the linearised transformation is

$$\Delta\Lambda_n = R'_*\Delta\Lambda_{n-1} \equiv 2^{-1/\tau}\Delta\Lambda_{n-1},$$

where $\tau = -\ln(2)/\ln(R'_*)$. Comparison with the definition of the bifurcation rate δ shows that $R'_* = \delta^{-1}$ and hence

$$\tau = \frac{\ln(2)}{\ln(\delta)}.$$

For Λ_n close to λ^*,

$$N(\Lambda_n) = 2N(\lambda^* + \Delta\Lambda_{n-1}).$$

Repeated application of this equation l times yields

$$\begin{aligned} N(\Lambda_n) &= 2^l N(\lambda^* + \Delta\Lambda_{n-l}) \\ &= 2^l N(\lambda^* + 2^{l/\tau}\Delta\Lambda_n), \end{aligned}$$

where the definition of τ is used in the last step. Since l is arbitrary, it can be chosen as $2^{l/\tau} \approx (\Delta\Lambda_n)^{-1}$ for Λ_n close to λ^*. Then one gets

$$N(\Lambda_n) \sim (\Delta\Lambda_n)^{-\tau},$$

and τ is similar to the correlation length exponent.

The transformation relating Λ_n and Λ_{n-1} can be obtained in the following way. The derivative of $f_{2^n}(x, \Lambda_n)$ at $x^*_{2^n 1}$ is -1. So an equation relating Λ_n and Λ_{n+1} can be obtained by equating the slopes of $f_{2^n}(x, \Lambda_n)$ and $f_{2^{n+1}}(x, \Lambda_{n+1})$ at the respective fixed points. Taking $n = 0$, the slope of $f(x, \Lambda_0)$ at $x^*_1 = 1 - \Lambda_0^{-1}$ is

$$f'(x^*, \Lambda_0) = 2 - \Lambda_0.$$

The slope of $f_2(x, \Lambda_1)$ at the fixed points x^*_{2k} can be computed as

$$f_2'(x^*_{2k}, \Lambda_1) = f'(x^*_{21}, \Lambda_1) f'(x^*_{22}, \Lambda_1) = 4 + 2\Lambda_1 - \Lambda_1^2.$$

Hence the transformation is

$$4 + 2\Lambda_1 - \Lambda_1^2 = 2 - \Lambda_0,$$

with a fixed point value $\lambda^* \approx 3.5615$. The derivative at λ^* is $R'_* \approx 0.1952$ and hence $\delta \approx 5.1224$. These results are first approximations to the numerically computed values quoted earlier. In fact, using the derivatives of $f_2(x, \Lambda_1)$ and $f_4(x, \Lambda_2)$ one gets $\lambda^* \approx 3.5702$ and $\delta \approx 4.6142$ which are much better approximations.

8.5 More Applications

The RG theory has played a major role in providing a clear understanding of two important problems which are introduced in this section. As the details of the calculations are somewhat lengthy, only the basic physics and usefulness of RG concepts are discussed here.

Kondo Problem

The first application is to the Kondo problem which describes the interaction of a single magnetic impurity in a metal with the conduction electrons. The magnetic impurity has an intrinsic spin and so it can interact with the

conduction electrons via the exchange effect. For higher temperatures such that $k_BT \gg |J|$, the coupling energy, one would expect the magnetic susceptibility, χ, due to the impurity to vary as $1/T$ according to Curie's law. Then the pertinent question is whether this behaviour is continued all the way down to very low temperatures or not. The experimental observation is that for anti-ferromagnetic interaction ($J < 0$), χ is nearly constant for $T < T_k$, a characteristic temperature called Kondo temperature. It is also found that the electrical resistivity increases below T_k in contrast to the usual decrease observed in metals. For ferromagnetic interaction ($J > 0$), χ is found to follow Curie's law and there is no anomalous increase in resistance at low temperatures.

These observations may be understood qualitatively in the following manner. With anti-ferromagnetic interaction, a conduction electron and the impurity can form a singlet (total spin zero) ground state. Then for k_BT less than the ground state energy E_0, the electron and the impurity behaves together as a single entity with no intrinsic magnetic moment and so χ does not show any temperature dependence. The Kondo temperature may, thus, be roughly defined as $k_BT \approx E_0$. On the other hand, if the interaction is ferromagnetic, the ground state of the electron-impurity pair is a triplet state (total spin 1) and then the combined system behaves according to Curie's law. The many body aspect of the problem becomes evident if one considers the scattering events caused by the exchange effect leading to a flipping of the electron spin. If the impurity spin is down before the scattering, a spin up electron can be scattered to the spin down state via the anti-ferromagnetic interaction there by making the impurity spin up. But now, another spin up electron can not be scattered since the interaction is ineffective. Thus the conduction electrons can not be treated independently and so the problem has the many body character. For low temperatures, conduction electrons with energies in the neighbourhood of the Fermi surface alone are involved in the process, however, there is a continuum of energy states and thus multiple energy scales are involved. Perturbation theory calculations show that the scattering amplitude varies as $J\rho + (J\rho)^2 \ln(E_c/K_BT) + \cdots$ where ρ is the density of electron states and E_c is a cut-off energy. For low temperatures, the second term diverges logarithmically, and the perturbation theory becomes invalid. A possible way to tackle the problem is to group together electrons according to their energy and account for their interaction in a recursive manner by considering one group at a time. A quantitative analysis of the problem was lacking till Wilson's work using the RG approach which clearly established that the magnetic susceptibility due to the impurity is nearly constant below the Kondo temperature.

Turbulence

The second application which needs to be mentioned is to the description of turbulence in fluid flow. During the 1880s, O. Reynolds showed that the flow of an incompressible fluid, like water, through a pipe becomes turbulent for flow velocities greater than a critical value which depends on the type of the fluid and the pipe diameter. For velocities lower than the critical value, fluid flow is of the laminar type. The turbulent nature of the flow indicates the presence of random velocity fields (in space, time and direction) over a mean macroscopic flow velocity. The parameter characterizing the transition from laminar to turbulent flow is called the Reynolds number defined as $R = vl/\nu$ where v is the macroscopic flow speed, l is the typical size of the pipe and ν is the fluid viscosity. Reynolds' experiments showed that the minimum value of R required to induce turbulent flow is around 2000. The simplest of all turbulent flows is that in a region of space away from boundaries as occurring in the atmosphere or the ocean. The problem becomes still simpler if the average flow properties are assumed to be spatially uniform, isotropic and stationary. In this limiting case, the two point correlation function of the velocity field components, $< u_\alpha(\mathbf{r_1}, \mathbf{t_1})\mathbf{u}_\beta(\mathbf{r_2}, \mathbf{t_2}) >$, depends only on $|\mathbf{r_1} - \mathbf{r_2}|$ and $t_1 - t_2$, and the mean square turbulent energy $\bar{E} =< u^2(\mathbf{r}, \mathbf{t}) >$ is independent of $\mathbf{r}$ and t. A concept of central importance in the physics of turbulence is the energy (or wavenumber) spectrum $E(k)$ which is the Fourier transform of the correlation function $< \mathbf{u}(\mathbf{0}, \mathbf{t}) \cdot \mathbf{u}(\mathbf{r}, \mathbf{t}) >$. The turbulent energy can then be expressed as $\bar{E} = \int E(k)dk$. Thus $E(k)dk$ represents the energy distributed in the random velocity fields with wavenumbers in the shell between k and $k + dk$. $E(k)$ is analogous to the Fourier transform of the correlation function of spin density.

The Navier-Stokes equation (NSE) which describes the space-time evolution of the velocity field in an incompressible, Newtonian fluid is intrinsically nonlinear. The nonlinear terms arise out of the convective derivative in the momentum conservation equation, and also from the elimination of pressure gradients using the continuity equation. In addition, the NSE has a dissipative term of diffusive type resulting out of the Newtonian form for the stress tensor. The NSE is more clearly analyzed when expressed in terms of the Fourier components of the velocity field. This also makes the calculation of the energy spectrum more easier. The Fourier representation of the NSE shows that the velocity component with wavevector $\mathbf{k_1}$ is nonlinerly driven by the sum of interactions from other components with wavevectors $\mathbf{k_2}$ and $\mathbf{k_3}$ such that $\mathbf{k_1} = \mathbf{k_2} + \mathbf{k_3}$. This nonlinear coupling between the modes leads to transfer of energy from the modes with small k to those with larger k where viscous dissipation is predominant. Viscous damping is rapid when velocity gradients

are maximum and hence it occurs at small length scales (or large k) of the order of molecular mean free path. In fact the linear decay rate of a mode with wavenumber k is found to be νk^2. Thus turbulent flow is characterized by a large range of wavenumbers, the smallest and largest limits are decided by the macroscopic size of the flow and Reynolds number respectively. Nonlinear coupling of modes destroys the dependence of energy distribution in the higher k region on the manner in which turbulence is generated. This aspect makes the energy spectrum for large k to have a universal form. Formulation of these ideas led Kolmogorove (1941) to propose that the energy spectrum varies as $E(k) \sim k^{-5/3} f(k/k_d)$ where k_d is the maximum wavenumber present in the flow pattern and f is a universal function. Theoretical justification of this scaling form and the calculation of f has been one of the major aims of turbulence physics. The RG ideas for dynamical critical phenomena have been successfully employed for achieving this goal. The Fourier components of the velocity field are first divided into two groups, one for $0 < k \leq k_d/q$ and the other for $k_d/q < k \leq k_d$. Then the second group is eliminated from the NSE and the variables are rescaled as in all RG approaches. The viscosity constant, which is a parameter present in the NSE, is generalized to have a k dependence and redefined at every step of the RG transformation. The isolation of a fixed point viscosity function and the emergence of scaling behaviour has been established in this manner.

8.6 Concluding Remarks

The applications of RG theory discussed in this chapter are not at all exhaustive. The principal aim has been to introduce problems in diverse fields of physics which have been successfully analyzed with the RG formalism. In fact there are more applications to systems which have (i) frozen random properties, (ii) long range interactions, (iii) anisotropic coupling constants, (iv) quantum effects, etc. Yet another area is the field of particle diffusion on random and fractal models of disorder. Discussion of all these topics is beyond the scope of this introductory monograph. So it is concluded here with a few remarks on the origins of the RG approach in the field theories of elementary particles.

The technique of renormalization was developed in the 1950s to circumvent the problem of ultraviolet divergences in relativistic field theories (RFT) such as quantum electrodynamics. It is associated with great names such as Dyson, Feynman, Schwinger, Gell-mann, Weinberg and several others. The framework of RFTs (in four dimensional space-time) is similar to that of statistical mechanics in four ($d = 4$) dimension. The formulation of these theories is a generalization of Feynman's path integral approach to quantum mechan-

ics which begins with a definition of the probability amplitude $K(x_1, t_1|x_2, t_2)$ as a functional integral involving the action functional $A[x]$. In field theories, the action is defined as the space-time integral of a Lagrangian density. The central differences between the formulations of field theories and statistical mechanics are, (i) the partition function is replaced by a generating functional and (ii) the Boltzmann factor $\exp(-H)$ is changed to $\exp(\imath A)$. A generating function for calculating the averages and correlation functions of spin density was introduced in the section on perturbation theory in Chapter 2. The Lagrangian density of interacting fields contain parameters like particle mass m and coupling constant u describing the particle-field interaction. These parameters are similar to a_2 and a_4 in the Landau-Ginzburg hamiltonian. The Greens functions (or propagators), which yield the scattering amplitudes, are quantities of primary interest in RFTs. For instance, the two point Greens function $G_2(\mathbf{p}_1, \mathbf{p}_2)$ provides the amplitude for scattering of a state with momentum $\mathbf{p}_1$ to $\mathbf{p}_2$. Similarly, there are higher order Greens functions G_4, G_6, etc. These are analogous to the correlation functions in statistical mechanics, and can be obtained from the generating functional. In RFTs, there is no cut-off momentum similar to the cut-off wavevector Λ in statistical models. When the Greens functions are calculated using a perturbation expansion in the coupling constant u, the successive approximations are found to diverge in the high momentum (or wavevector) limit Λ. (Such divergences are evident in the perturbation expansion for the partition function discussed in Chapter 2.) If a cut-off momentum Λ is introduced, one finds that G_2 diverges as Λ^2 (when $d = 4$) while G_4 varies as $\ln(\Lambda)$. The renormalization program was invented to remove these higher momentum or ultraviolet divergences. One prescription is to define renormalized parameters m_r and u_r in terms of values of G_2 and G_4 when $\sum_i \mathbf{p}_i = 0$. Then all the Greens functions are expressed in terms of m_r, u_r and Λ. If it is possible to choose these parameters such that the Greens functions are finite in the limit $\Lambda \to \infty$ keeping m_r and u_r fixed, then the theory is said to be renormalizable. The parameters m_r and u_r are then taken as the observed values of these quantities.

The RG in this formalism arose from the observation that the momentum chosen to define the parameters m_r and u_r is arbitrary. So one may define them when the net momentum is not zero but has some value $\mathbf{p}_0$. Then it turns out that a particular theory can be expressed in terms of a family of renormalized parameters which satisfy equations analogous to the transformation equations for the parameters in H. This is the origin of the RG concept. As Wilson remarks, this idea of renormalization does not have any physical basis. It simply expresses the arbitrariness in the definition of the renormalized parameters. Further, it is based entirely on the perturbative approach for calculations. The problem in critical phenomena is the infrared divergence

of correlation functions near the critical point. There is always a physically meaningful cut-off wavevector Λ, so the ultraviolet divergences do not exist. Wilson's generalization of the RG approach involved transforming all the parameters in the hamiltonian and so the hamiltonian itself. The new approach, thus, can be seen as a theory for generating newer models and the method of generation is the physical process of coarse graining the system which has to be chosen depending on the problem at hand. The new models generated turn out to be 'simpler' for extracting the common features observed in a class of phenomena. Thus the RG approach is much more than a technique for calculations in phase transition theory and has the potential to unify the physics underlying many complex systems. The applications outlined in this chapter provide only a flavor of this potential.

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About the book

In nature there are several phenomena like thermal phase transitions or percolation processes which involve a multitude of length scales and / or time scales. For describing such phenomena, Kenneth Wilson, around 1970, put forward the renormalization group theory. The basic ideas and techniques of the theory are elaborated in this monograph using some simple models of ferromagnetic critical behaviour. Brief outlines of applications to some of the related areas are also given. This monograph would provide a self contained introduction to beginners.

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